

LAPLACE EQUATION. The linear map $T \mapsto \Delta T = T_{xx} + T_{yy}$ for smooth functions in the plane is called the **Laplacian**. The PDE $\Delta T = 0$ is called the **Laplace equation**. A solution T is in the kernel of Δ and called **harmonic**. For example $T(x, y) = x^2 - y^2$ is harmonic. One can show that $T(x, y) = \operatorname{Re}((x + iy)^n)$ or $T(x, y) = \operatorname{Im}((x + iy)^n)$ are all harmonic.

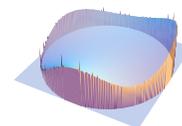
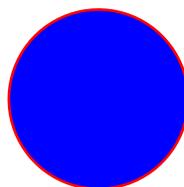
DIRICHLET PROBLEM: find a function $T(x, y)$ on a region Ω which satisfies $\Delta T(x, y) = 0$ inside Ω and which is prescribed function $f(x, y)$ at the boundary. This **boundary value problem** can be solved explicitly in many cases.

IN A DISC. $\{x^2 + y^2 \leq 1\}$. If $f(t) = T(\cos(t), \sin(t))$ is prescribed on the boundary and T satisfies $\Delta T = 0$ inside the disc, then T can be found via Fourier theory:

If $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$, then $T(x, y) = a_0 + \sum_n a_n \operatorname{Re}((x + iy)^n) + b_n \operatorname{Im}((x + iy)^n)$ satisfies the Laplace equation and $T(\cos(t), \sin(t)) = f(t)$ on the boundary of the disc.

PROOF. If $f = a_0$ is constant on the boundary then $T(x, y) = a_0$. If $f(t) = \sin(nt)$ on the boundary $z = e^{it} = x + iy$ then $T(x, y) = \operatorname{Im}(z^n)$ and if $f(t) = \cos(nt)$ on the boundary then $T(x, y) = \operatorname{Re}(z^n)$ is a solution.

EXAMPLE. $\operatorname{Re}(z^2) = x^2 - y^2 + i(2xy)$ gives the two harmonic functions $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$: they satisfy $(\partial_x^2 + \partial_y^2)f = 0$.



ESPECIALLY: the **mean value property for harmonic functions** $T(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$ holds.

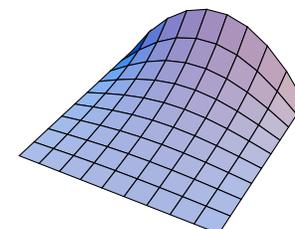
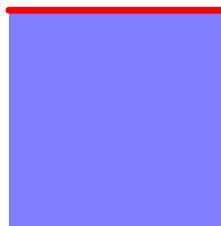
POISSON FORMULA: With $z = x + iy = re^{i\theta}$ and $T(x, y) = f(z)$, the general solution is given also by the **Poisson integral formula**:

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{r^2 - 2r \cos(\theta - \phi) + 1} f(e^{i\phi}) d\phi.$$

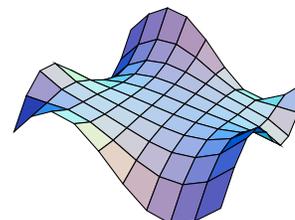
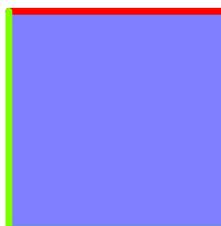
PHYSICAL RELEVANCE. If the **temperature** or **charge** distribution on the boundary of a region Ω , then $T(x, y)$ is the stable temperature or **charge** distribution inside that region. In the later case, the electric field inside Ω is then ∇U . An other application is that the velocity v of an ideal incompressible fluid satisfies $v = \nabla U$, where U satisfies the Laplace differential equation $\Delta U = 0$ in the region.

SQUARE, SPECIAL CASE. We solve first the case, when T vanishes on three sides $x = 0, x = \pi, y = 0$ and $T(x, \pi) = f(x)$. Separation of variables gives then $T(x, y) = \sum_n a_n \sin(nx) \sinh(ny)$, where the coefficients a_n are obtained from $T(x, \pi) = \sum_n b_n \sin(nx) = \sum_n a_n \sin(nx) \sinh(n\pi)$, where $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx$ are the Fourier coefficients of f . The solution is therefore

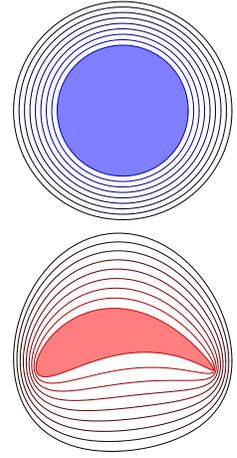
$$T_f(x, y) = \sum_{n=1}^{\infty} b_n \sin(nx) \sinh(ny) / \sinh(n\pi)$$



SQUARE GENERAL CASE Solutions to general boundary conditions can be obtained by adding the solutions in the following cases: $T_f(x, \pi - y)$ solves the case, when T vanishes on the sides $x = 0, x = \pi, y = \pi$ and is f on $y = 0$. The function $T_f(y, x)$ solves the case, when T vanishes on the sides $y = 0, y = \pi, x = 0$ and $T_f(\pi - y, x)$ solves the case, when T vanishes on $y = 0, y = \pi$ and $x = \pi$. The general solution is $T(x, y) = T_f(x, y) + T_g(x, \pi - y) + T_h(y, x) + T_k(\pi - y, x)$



A GENERAL REGION. For a general region, one uses numerical methods. One possibility is by **conformal transportation**: to map the region into the disc using a complex map F , which maps the region Ω into the disc. Solve then the problem in the disc with boundary value $T(F^{-1}(x, y))$ on the disc.

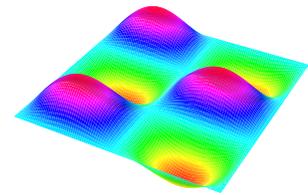
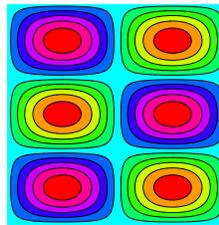


If $S(x, y)$ is the solution there, then $T(x, y) = S(F^{-1}(x, y))$ is the solution in the region Ω . The picture shows the example of the Joukowski map $z \mapsto (z + 1/z)/2$ which has been used in fluid dynamics (N.J. Joukowski (1847-1921) was a Russian aerodynamics researcher.)

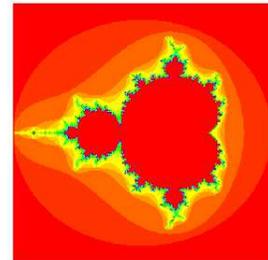
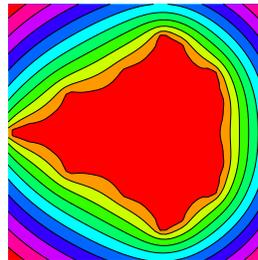
A tourists view on related PDE topics

POISSON EQUATION. The Poisson equation $\Delta f = g$ in a square $[0, \pi]^2$ can be solved via Fourier theory: If $f(x, y) = \sum_{k,m} a_{n,m} \cos(nx) \cos(my)$ and $g(x, y) = \sum_{k,m} b_{n,m} \cos(nx) \cos(my)$, then $f_{xx} + f_{yy} = \sum_{n,m} -(n^2 + m^2)a_{n,m} \cos(nx) \cos(my) = \sum_{k,m} b_{n,m} \cos(nx) \cos(my)$, then $a_{n,m} = -b_{n,m}/(n^2 + m^2)$. The Poisson equation is important in **electrostatics**, for example to determine the electromagnetic field when the charge and current distribution is known: $\Delta U(x) = -(1/\epsilon_0)\rho$, then $E = \nabla U$ is the electric field to the charge distribution $\rho(x)$.

EIGENVALUES. The functions $\sin(nx) \sin(ny)$ are eigenfunctions for the Laplace operator $f_{xx} + f_{yy} = n^2 f$ on the disc. For a general bounded region Ω , we can look at all smooth functions which are zero on the boundary of Ω . The possible eigenvalues of $\Delta f = \lambda f$ are the possible energies of a particle in Ω .

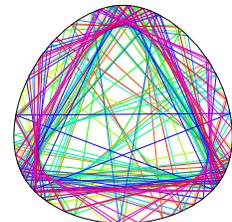


COMPLEX DYNAMICS. Also in complex dynamics, harmonic functions appear. Finding properties of complicated sets like the **Mandelbrot set** is done by mapping the exterior to the outside of the unit circle. If the Mandelbrot set is charged then the contour lines of equal potential can be obtained as the corresponding contour lines in the disc case (where the lines are circles).



SCHRÖDINGER EQUATION. If H is the energy operator, then $i\hbar \dot{f} = Hf$ is called the **Schrödinger equation**. If $H = -\hbar^2 \Delta / (2m)$, this looks very much like the heat equation, if there were not the i . If f is an eigenvalue of H , then $i\hbar \dot{f} = \lambda f$ and $f(t) = e^{i\lambda t} f(0)$. In the heat equation, we would get $f(t) = e^{-\mu(t)} f(0)$. The evolution of the Schrödinger equation is very similar to the wave equation.

QUANTUM CHAOS studies the eigenvalues and eigenvectors of the Laplacian in a bounded region. If the billiard in the region is chaotic, the study of the eigensystem is called **quantum chaos**. The eigenvalue problem $\Delta f = \lambda f$ and the **billiard problem** in the region are related. A famous open problem is whether two smooth convex regions in the plane for which the eigenvalues λ_j are the same are equivalent up to rotation and translation.



In a square of size L we know all the eigenvalues $\frac{\pi^2}{L^2}(n^2 + m^2)$. The eigenvectors are $\sin(\frac{\pi}{L}nx) \sin(\frac{\pi}{L}mx)$. We have explicit formulas for the eigenfunctions and eigenvalues. On the classical level, for the billiard, we have also a complete picture, how a billiard path will look like. The square is an example, where we have no quantum chaos.

