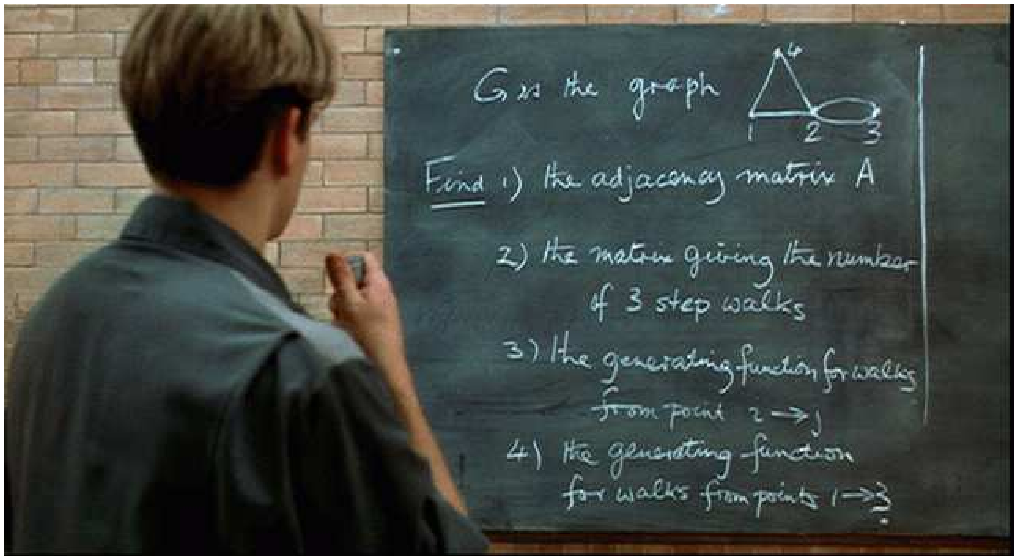
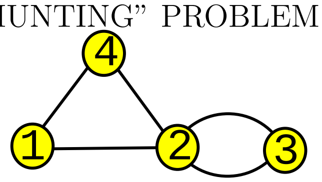
ICE: A BLACKBOARD PROBLEM **Math 21b, O. Knill**

In the movie "Good Will Hunting", the main character Will Hunting (Matt Damon) solves a blackboard challenge problem, which is given as a challenge to a linear algebra class.



THE "WILL HUNTING" PROBLEM.

G is the graph



<u>Find</u>.
1) the adjacency matrix A.
2) the matrix giving the number of 3 step walks.
3) the generating function for walks from point $i \to j$.
4) the generating function for walks from points $1 \to 3$.

This problem belongs to linear algebra and calculus even so the problem origins from graph theory or combinatorics. For a calculus student who has never seen the connection between graph theory, calculus and linear algebra, the assignment is actually hard – probably too hard - as the movie correctly indicates. The problem was posed in the last part of a linear algebra course. An explanation of some terms:

THE ADJACENCY MATRIX. The structure of the graph can be encoded with a 4×4 array which encodes how many paths of length 1, one can take in the graph from one node to an other:

no	one	no	one
one	none	two	one
no	two	no	no
one	one	no	no

which can more conveniently be written as an array of numbers called a **matrix**:

$$L = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Problem 2 asks to find the matrix which encodes all possible paths of length 3.

GENERATING FUNCTION. To any graph one can assign for every pair of nodes i, j a series $f(z) = \sum_{n=0}^{\infty} a_n^{(ij)} z^n$, where $a_n^{(ij)}$ is the number of possible walks from node i to node j with n steps. Problem 3) asks for an explicit expression of $f(z)$ and problem 4) asks for an explicit expression in the special case $i = 1, j = 3$.

Linear algebra has many relations to other fields in mathematics. It is not true that linear algebra is just about solving systems of linear equations.

SOLUTION TO 2). $[L^2]_{ij}$ is by definition of the matrix product the sum $L_{i1}L_{1j} + L_{i2}L_{2j} + \ldots + L_{i4}L_{hj}$. Each term $L_{ik}L_{kj}$ is 1 if and only if there is a path of length 2 going from i to j passing through k. Therefore $[L^2]_{ij}$ is the number of paths of length 2 going from node i to j. Similarly, $[L^n]_{ij}$ is the number of paths of length n going from i to j. The answer is

$$L^3 = \begin{bmatrix} 2 & 7 & 2 & 3 \\ 7 & 2 & 12 & 7 \\ 2 & 12 & 0 & 2 \\ 3 & 7 & 2 & 2 \end{bmatrix}$$

SOLUTION TO 3). The geometric series formula

$$\sum_n x^n = (1-x)^{-1}$$

holds also for matrices:

$$f(z) = \sum_{n=0}^{\infty} [L^n]_{ij} z^n = [\sum_{n=0}^{\infty} L^n z^n)]_{ij} = (1 - Lz)^{-1}$$

Cramer's formula for the inverse of a matrix which involves the determinant using the adjugate matrix $[A]$ which is the transpose of the minors multiplied by $(-1)^{i+j}$:

$$A^{-1} = \mathrm{adj}(A)/\det(A) \; .$$

This leads to an explicit formula $(-1)^{i+j}\det(1 - z[L]_{ji})/\det(1 - zL)$.

SOLUTION TO 4).

$$(1 - Lz)^{-1} = \begin{bmatrix} \frac{1-5z^2}{4z^4-2z^3-7z^2+1} & \frac{z^2+z}{4z^4-2z^3-7z^2+1} & \frac{2z^3+2z^2}{4z^4-2z^3-7z^2+1} & \frac{-4z^3+z^2+z}{4z^4-2z^3-7z^2+1} \\ \frac{z^2+z}{4z^4-2z^3-7z^2+1} & \frac{1-z^2}{4z^4-2z^3-7z^2+1} & \frac{2z-2z^3}{4z^4-2z^3-7z^2+1} & \frac{z^2+z}{4z^4-2z^3-7z^2+1} \\ \frac{2z^3+2z^2}{4z^4-2z^3-7z^2+1} & \frac{2z-2z^3}{4z^4-2z^3-7z^2+1} & \frac{-2z^3-3z^2+1}{4z^4-2z^3-7z^2+1} & \frac{2z^3+2z^2}{4z^4-2z^3-7z^2+1} \\ \frac{-4z^3+z^2+z}{4z^4-2z^3-7z^2+1} & \frac{z^2+z}{4z^4-2z^3-7z^2+1} & \frac{2z^3+2z^2}{4z^4-2z^3-7z^2+1} & \frac{1-5z^2}{4z^4-2z^3-7z^2+1} \end{bmatrix}$$

The entry $i = 1$ and $j = 3$ is

$$(1 - zL)^{-1}_{13} = \frac{2z^3 + 2z^2}{4z^4 - 2z^3 - 7z^2 + 1}$$

because

$$\mathrm{adj}(1 - zL)_{13} = (-1)^{1+3}\det(\begin{bmatrix} -z & 1 & -z \\ 0 & -2z & 0 \\ -z & -z & 1 \end{bmatrix}) = 2z^2 + 2z^3$$

and

$$\det(1 - zL) = \det \begin{bmatrix} 1 & -z & 0 & -z \\ -z & 1 & -2z & -z \\ 0 & -2z & 1 & 0 \\ -z & -z & 0 & 1 \end{bmatrix} = 4z^4 - 2z^3 - 7z^2 + 1 \; .$$