CALCULATING EIGENVECTORS — Math 21b, O.Knill

NOTATION. We often just write 1 instead of the identity matrix 1_n.

COMPUTING EIGENVALUES. Recall: because $\lambda - A$ has $\vec{v}$ in the kernel if λ is an eigenvalue the characteristic polynomial $f_A(\lambda) = \det(\lambda - A) = 0$ has eigenvalues as roots.

2×2 CASE. Recall: The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)/2\lambda + (ad - bc)$. The eigenvalues are $\lambda_\pm = T/2 \pm \sqrt{(T/2)^2 - D}$, where $T = a + d$ is the trace and $D = ad - bc$ is the determinant of A. If $(T/2)^2 \geq D$, then the eigenvalues are real. Away from that parabola in the (T, D) space, there are two different eigenvalues. The map A contracts volume for $|D| < 1$.

NUMBER OF ROOTS. Recall: There are examples with no real eigenvalue (i.e. rotations). By inspecting the graphs of the polynomials, one can deduce that $n \times n$ matrices with odd n always have a real eigenvalue. Also $n \times n$ matrixes with even n and a negative determinant always have a real eigenvalue.

IF ALL ROOTS ARE REAL. $f_A(\lambda) = \lambda^n - \text{tr}(A)\lambda^{n-1} + ... + (-1)^n \det(A) = (\lambda - \lambda_1)...(\lambda - \lambda_n)$, we see that $\sum_i \lambda_i = \text{trace}(A)$ and $\prod_i \lambda_i = \det(A)$.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)\vec{v} = 0$, the vector $\vec{v}$ is in the kernel of $\lambda - A$.

EIGENVECTORS of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\vec{v}_\pm$ with eigenvalue $\lambda_\pm$.

If $c = 0$ and $d = 0$, then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors.

If $c \neq 0$, then the eigenvectors to $\lambda_\pm$ are $\vec{v}_\pm = \begin{bmatrix} \lambda_\pm - d \\ c \end{bmatrix}$.

If $c = 0$ and $a \neq d$, then the eigenvectors to a and d are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} b/(d-a) \\ 1 \end{bmatrix}$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$, then f has **algebraic multiplicity** k. The algebraic multiplicity counts the number of times, an eigenvector occurs.

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

GEOMETRIC MULTIPLICITY. The dimension of the eigenspace E_λ of an eigenvalue λ is called the **geometric** multiplicity of λ.

EXAMPLE: the matrix of a shear is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has the eigenvalue 1 with algebraic multiplicity 2. The kernel of $A - 1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the geometric multiplicity is 1. It is different from the algebraic multiplicity.

EXAMPLE: The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalue 1 with algebraic multiplicity 2 and the eigenvalue 0 with multiplicity 1. Eigenvectors to the eigenvalue $\lambda = 1$ are in the kernel of $A - 1$ which is the kernel of $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The geometric multiplicity is 1.

RELATION BETWEEN ALGEBRAIC AND GEOMETRIC MULTIPLICIY. (Proof later in the course). The geometric multiplicity is smaller or equal than the algebraic multiplicity.

PRO MEMORIA. Remember that the **geometric mean** $\sqrt{ab}$ of two numbers is smaller or equal to the **algebraic mean** $(a + b)/2$? (This fact is totally* unrelated to the above fact and a mere coincidence of expressions, but it helps to remember it). Quite deeply buried there is a connection in terms of convexity. But this is rather philosophical. .

EXAMPLE. What are the algebraic and geometric multiplicities of $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$?

SOLUTION. The algebraic multiplicity of the eigenvalue 2 is 5. To get the kernel of $A-2$, one solves the system of equations $x_4 = x_3 = x_2 = x_1 = 0$ so that the geometric multiplicity of the eigenvalues 2 is 4.

CASE: ALL EIGENVALUES ARE DIFFERENT.

If all eigenvalues are different, then all eigenvectors are linearly independent and all geometric and algebraic multiplicities are 1.

PROOF. Let λ_i be an eigenvalue different from 0 and assume the eigenvectors are linearly dependent. We have $v_i = \sum_{j \neq i} a_j v_j$ and $\lambda_i v_i = A v_i = A(\sum_{j \neq i} a_j v_j) = \sum_{j \neq i} a_j \lambda_j v_j$ so that $v_i = \sum_{j \neq i} b_j v_j$ with $b_j = a_j \lambda_j / \lambda_i$.
If the eigenvalues are different, then $a_j \neq b_j$ and by subtracting $v_i = \sum_{j \neq i} a_j v_j$ from $v_i = \sum_{j \neq i} b_j v_j$, we get $0 = \sum_{j \neq i} (b_j - a_j) v_j = 0$. Now $(n-1)$ eigenvectors of the n eigenvectors are linearly dependent. Use induction.

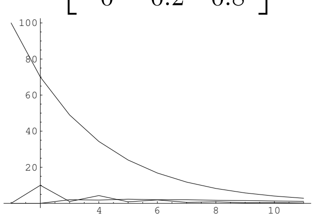
CONSEQUENCE. If all eigenvalues of a $n \times n$ matrix A are different, there is an **eigenbasis**, a basis consisting of eigenvectors.

EXAMPLE. $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ has eigenvalues $1, 3$ to the eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

EXAMPLE. (See homework problem 40 in the book).

Photos of the Swiss lakes in the text. The pollution story is fiction fortunately.

The vector $A^n(x)b$ gives the pollution levels in the three lakes (Silvaplana, Sils, St Moritz) after n weeks, where $A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$ and $b = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ is the initial pollution.



There is an eigenvector $e_3 = v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to the eigenvalue $\lambda_3 = 0.8$.

There is an eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ to the eigenvalue $\lambda_2 = 0.6$. There is further an eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ to the eigenvalue $\lambda_1 = 0.7$. We know $A^n v_1, A^n v_2$ and $A^n v_3$ explicitly.
How do we get the explicit solution $A^n b$? Because $b = 100 \cdot e_1 = 100(v_1 - v_2 + 3v_3)$, we have

$$\begin{aligned} A^n(b) &= 100 A^n(v_1 - v_2 + 3v_3) = 100(\lambda_1^n v_1 - \lambda_2^n v_2 + 3\lambda_3^n v_3) \\ &= 100\left(0.7^n \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 0.6^n \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 3 \cdot 0.8^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 100(0.7)^n \\ 100(0.7^n + 0.6^n) \\ 100(-2 \cdot 0.7^n - 0.6^n + 3 \cdot 0.8^n) \end{bmatrix} \end{aligned}$$