SOLUTIONS TO SECOND HOURLY **Math 21a, Spring 2003**

Name:

MWF10 Ken Chung
MWF10 Weiyang Qiu
MWF11 Oliver Knill
TTh10 Mark Lucianovic
TTh11.5 Ciprian Manolescu

- Start by printing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit. Except for T/F or matching problems, provide details how you obtained the solution.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1)	(20 points)

The solution key is | T T F F F F T F T F T F T F T T F T T T |.

| T | F | Given a function $f(x,y)$, then $F(x,y) = (f_x(x,y), f_x(x,y) + f_y(x,y))$ defines a vector field in the plane.

True: both coordinates P, Q of $F = (P, Q)$ are functions of two variables.

| T | F | Every smooth function of three variables $f(x,y,z)$ satisfies the partial differential equation $f_{xyz} + f_{yzx} = 2f_{zxy}$.

True: by Clairot's theorem.

| T | F | If $f_x(x,y) = f_y(x,y)$ for all x, y, then $f(x,y)$ is a constant.

False: $f_x = f_y$ is an example of a PDE called a transport equation. It has solutions like for example $f(x,y) = x + y$. Any function which stays invariant by replacing x with y is a solution: like $f(x,y) = \sin(xy) + x^5y^5$.

| T | F | $(1,1)$ is a local maximum of the function $f(x,y) = x^2y - x + \cos(y)$.

False: $(1,1)$ is not even a critical point.

| T | F | If f is a smooth function of two variables, then the number of critical points of f inside the unit disc is finite.

False: Take $f(x,y) = x^2$ for example. Every point on the y axes $\{x = 0\}$ is a critical point.

| T | F | The value of the function $f(x,y) = \sin(-x + 2y)$ at $(0.001, -0.002)$ can by linear approximation be estimated as -0.003.

False: The correct approximation would be $f(0,0) + 0.001(-1) - 0.002(2) = -0.005$.

| T | F | If $(1,1)$ is a critical point for the function $f(x,y)$ then $(1,1)$ is also a critical point for the function $g(x,y) = f(x^2, y^2)$.

True: If $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (0,0)$ then also $\nabla g(1,1) = (f_x(1,1)2x, f_y(1,1)2y) = (0,0)$. Note that the statement would not be true, if we would replace $(1,1)$ say with $(2,2)$ (as in the practice exam).

| T | F | There is no function $f(x,y,z)$ of three variables, for which every point on the unit sphere is a critical point.

False: This is a 3D version of the problem in the practice exam: just take a function like $g(t) = te^{-t}$ with a maximum at $t = 1$ and define $f(x,y,z) = g(x^2 + y^2 + z^2)$.

| T | F | If the double integral $\int\int_R f(x,y)\, dxdy$ is zero for a continuous function $f(x,y)$ and R is the interior of the unit disc, then $f(x,y) = 0$ for at least one point x, y.

True: Assume the conclusion were false, then either $f(x,y) > 0$ everywhere on the disc, or $f(x,y) < 0$ everywhere on the disc. In both cases, the double integral would not vanish.

| T | F | The surface area is given by the formula $\int\int_R \vec{r}_u \times \vec{r}_v\, dudv$.

False: Take absolute values $|\vec{r}_u \times \vec{r}_v|$.

| T | F | The gradient of $f(x,y)$ is normal to the level curves of f.

True: A basic and important fact.

| T | F | If (x_0, y_0) is a maximum of $f(x,y)$ under the constraint $g(x,y) = g(x_0, y_0)$, then (x_0, y_0) is a maximum of $g(x,y)$ under the constraint $f(x,y) = f(x_0, y_0)$.

False: Assume you have a situation f, g, where this is true and where the constraint is $g = 0$, produce a new situation $f, h = -g$, where the first statement is still true but where the extrema of h under the constraint of f is a minimum.

| T | F | If $\vec{u}$ is a unit vector tangent at (x,y,z) to the level surface of $f(x,y,z)$ then $D_u f(x,y,z) = 0$.

True: The directional derivative measures the rate of change of f in the direction of u. On a level surface, in the direction of the surface, the function does not change (because f is constant by definition on the surface).

T	F	When changing to cylindrical coordinates (ρ, θ, ϕ), one has to include the Jacobean $\rho^2 \sin(\theta)$.

False: this statement is wrong in two ways. First of all, the cylindrical coordinates do not involve two angles, second, even if spherical coordinates were ment, the correct integration factor is $\rho^2 \sin(\phi)$ not $\rho^2 \sin(\theta)$.

T	F	The function $u(x,t) = x^2/2 + t$ satisfies the heat equation $u_t = u_{xx}$.

True: Just differentiate.

T	F	The vector $\vec{r}_u - \vec{r}_v$ is tangent to the surface parameterized by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$.

True: Both vectors $\vec{r}_u$ and $\vec{r}_v$ are tangent to the surface. So also their difference.

T	F	Fubini's theorem assures that $\int_0^1 \int_0^x f(x,y)\,dydx = \int_0^1 \int_0^y f(x,y)\,dxdy$.

False: Fubini is the corresponding statement, when the bounds of integration are constants, that is if you integrate over a rectangle.

T	F	If $(1,1,1)$ is a maximum of f under the constraints $g(x,y,z) = c, h(x,y,z) = d$, and the Lagrange multipliers satisfy $\lambda = 0, \mu = 0$, then $(1,1,1)$ is a critical point of f.

True: Look at the Lagrange equations. If $\lambda = \mu = 0$, then $\nabla f = (0,0,0)$.

T	F	If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant D is zero but $f_{xx}(0,0) > 0$ then $(0,0)$ can not be a local maximum.

True: If $f_{xx}(0,0) > 0$ then on the x-axes the function $g(x) = f(x,0)$ has a local minimum. This means that there are points close to $(0,0)$ where the value of f is larger.

T	F	Let (x_0, y_0) be a saddle point of $f(x,y)$. For any unit vector $\vec{u}$, there are points arbitrarily close to (x_0, y_0) for which ∇f is parallel to $\vec{u}$.

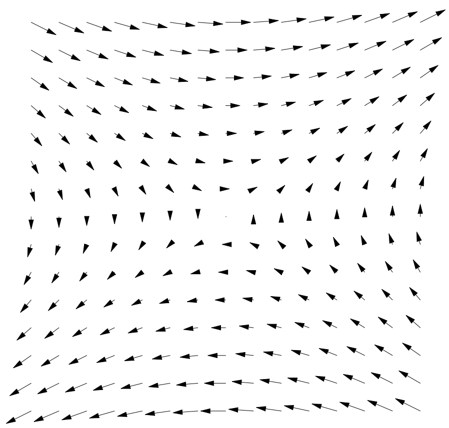
True: Just look at the level curves near a saddle point. The gadient vectors are orthogonal to the level curves which are hyperbola. You see that they point in any direction except 4 directions. To see this better, take a pen and draw a circle around the saddle point between two of your knuckles on your fist. At each point of the circle, now draw the direction of steapest increase (this is the gradient direction).

Problem 2) (10 points)

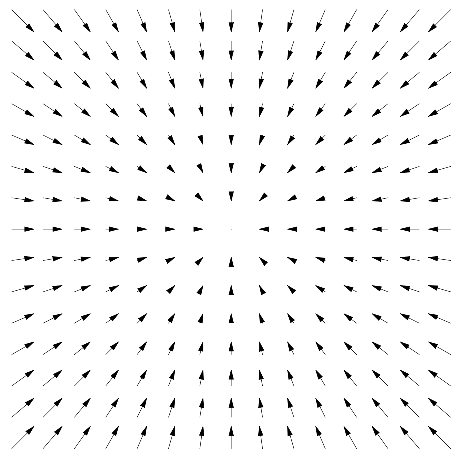
Match the formulas for the vectorfields with the corresponding picture I,II,III or IV.

Vectorfield	Enter I, II, III or IV
$F(x,y) = (y,0)$	IV
$F(x,y) = (3,5)$	III
$F(x,y) = (-x,-y)$	II
$F(x,y) = (2y,x)$	I

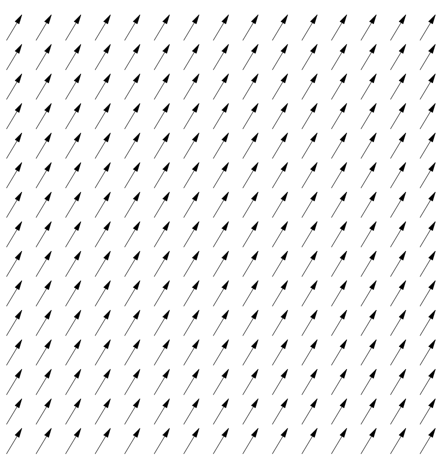
I



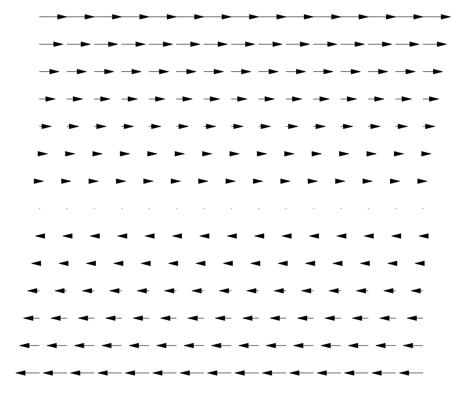
II



III



IV



Problem 3) (10 points)

Use the technique of linear approximation to estimate $f(\log(2)+0.001, 0.006)$ for $f(x,y) = e^{2x-y}$. (Here, log means the natural logarithm).

Solution. $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
$f(x_0,y_0) = e^{2\log 2} = 4$
$f_x(x_0,y_0) = 8$
$f_y(x_0,y_0) = -4$

Solution: $L(x, y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = \boxed{3.984}$.

Problem 4) (10 points)

a) Show that for any differentiable function $g(x)$, the function $u(x, y) = g(x^2 + y^2)$ satisfies the partial differential equation $yu_x = xu_y$.

b) Assuming $g'(5) \neq 0$, let u be the function defined in a). Find the unit vector $\vec{v}$ in the direction of maximal increase at the point $(x, y) = (2, 1)$.

Solution. a) Just differentiate:
$yu_x = yg'(x^2 + y^2)2x = 2xyg'(x^2 + y^2)$
$xu_y = xg'(x^2 + y^2)2y = 2yxg'(x^2 + y^2)$
These two expressions are the same.

b) The direction of maximal increase points into the direction of the gradient of u which is $\nabla u(x, y) = (g'(x^2 + y^2)2x, g'(x^2 + y^2)2y)$.
At the point $(x, y) = (2, 1)$ we have $(g'(5)4, g'(5)2)$. If we normalize that, we obtain $\boxed{\vec{v} = (4, 2)/\sqrt{20}}$.

Problem 5) (10 points)

Which point on the surface $g(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ is closest to the origin?

Solution. This is a Lagrange problem. One wants to minimize $f(x, y, z) = x^2 + y^2 + z^2$ under the constraint $g(x, y, z) = 1$. The Lagrange equations are

$$\begin{aligned} \frac{-1}{x^2} &= 2\lambda x \\ \frac{-1}{y^2} &= 2\lambda y \\ \frac{-8}{z^2} &= 2\lambda z \\ \frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1 \end{aligned}$$

The first two equations show $x = y$, the first and third equations show $8/z^3 = 1/x^3$ or $z = 2x$. Plugging this into the last equation gives $2/x + 8/(2x) = 1$ or $x = 6, y = 6, z = 12$.
$\boxed{(x, y, z) = (6, 6, 12)}$.

There is an interesting twist to this problem (as noted by one of the students Jacob Aptekar): consider the points $(x, y, z) = (1, -1/n, 8/n)$, where n is a large integer, One can check that these points ly on the surface $g(x, y, z) = 1$. Their distance to the origin however decreases to 1 if n goes to infinity. So the point $(6, 6, 12)$, while a local minimum is not a global minimum.

Problem 6) (10 points)

Find all extrema of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum amoung them?

Solution.
The critical points satisfy $\nabla f(x, y) = (0, 0)$ or $(3x^2 - 3, 3y^2 - 12) = (0, 0)$. There are 4 critical points $(x, y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x, 0) = x^3 - 3x + 20$ which satisfies $\lim_{x \to \pm\infty} g(x) = \pm\infty$.

Problem 7) (10 points)

Find

$$\int_0^1 \int_y^1 x^2 e^{xy} \, dxdy \, .$$

Hint. Sketch the region and check the order of integration.

Solution:

Integrate $f(x, y) = x^2 e^{xy}$ over the region, where $0 \leq y \leq 1$ and $y \leq x \leq 1$.

Switch the order of integration: $\int_0^1 \int_0^x x^2 e^{xy} \, dydx = \boxed{(e - 2)/2}$.

Problem 8) (10 points)

Integrate the function $f(x, y, z) = z^2$ over the region $\{x^2 + y^2 + z^2 \leq 1\}$.

Solution $\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \cos^2(\phi) \sin(\phi) \, d\rho d\phi d\theta = 2\pi(1/5)(2/3) = \boxed{4\pi/15}$.

Problem 9) (10 points)

Find the surface area of the surface $z = \sqrt{x^2 + y^2} + 1$ that lies above the unit disk $\{x^2 + y^2 \leq 1\}$ in the x-y plane.

Solution.
Write the surface in cylindrical coordinates $r(u, v) = (u \cos[v], u \sin[v], u + 1)$. Then $|r_u \times r_v| = \sqrt{2}u$, so that we have

$$\int_0^1 \int_0^{2\pi} \sqrt{2}u \, dvdu = 2\pi\sqrt{2}/2 \, .$$