SOLUTION FIRST HOURLY **Math 21a,Spring 2003**

Name:

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- Start by printing your name in the above box and check your section in the box to the left. ______
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points)

T	F	Here, the *'s are placed at the places with the correct answer.
*		The vector $\vec{v}$ connecting $(1,2,3)$ with $(4,5,3)$ is orthogonal to $\vec{w} = (0,0,3)$.
	*	The length of the difference $\vec{v} - \vec{w}$ of two parallel vectors is the difference $|\vec{v}| - |\vec{w}|$ of the lengths of the vectors.

Note: this is even false for numbers: $|a-b| \neq ||a|-|b||$ in general. A counterexample for vectors is $\vec{v} = (1,0,0)$ and $\vec{w} = (-1,0,0)$.

T	F	
	*	$\vec{k} = \vec{j} \times \vec{i}$.
*		The set of points in $\mathbf{R}^3$ which have distance 1 from a line form a cylinder.
	*	The equation $x^2 - z^2 = y$ describes a one-sheeted hyperboloid.
	*	If in rectangular coordinates, a point is given by $(1,0,1)$, then its spherical coordinates are $(\rho,\theta,\phi) = (\sqrt{2}, \pi/2, -\pi/2)$.
*		The curvature of a curve $\vec{r}(t)$ at $t=0$ is the same as the curvature of the curve $\vec{r}(-t)$ at $t=0$.
*		In spherical coordinates the equation $\cos(\theta) = \sin(\theta)$ defines the plane $x - y = 0$.
	*	The velocity vector of a parametric curve $\vec{r}(t)$ always has length 1.
*		In spherical coordinates (ρ,θ,ϕ), the equations $\phi = \pi/2 = \rho$ define a circle.
*		For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we always have $(\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{a} \times \vec{c}) \cdot \vec{b}$.
*		The set of points in the xy-plane which satisfy $x^2 - y^2 = -1$ is a hyperbola.
	*	If $|\vec{v} \times \vec{w}| = 0$ then $\vec{v} = 0$ or $\vec{w} = 0$.
*		Two nonzero vectors are parallel if and only if their cross product is $\vec{0}$.
*		If the velocity vector $\vec{r}\,'(t)$ and the acceleration vector $\vec{r}\,''(t)$ of a curve are parallel then the curvature $\kappa(t)$ of the curve is zero.
	*	The function $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is discontinuous at $(0,0)$.

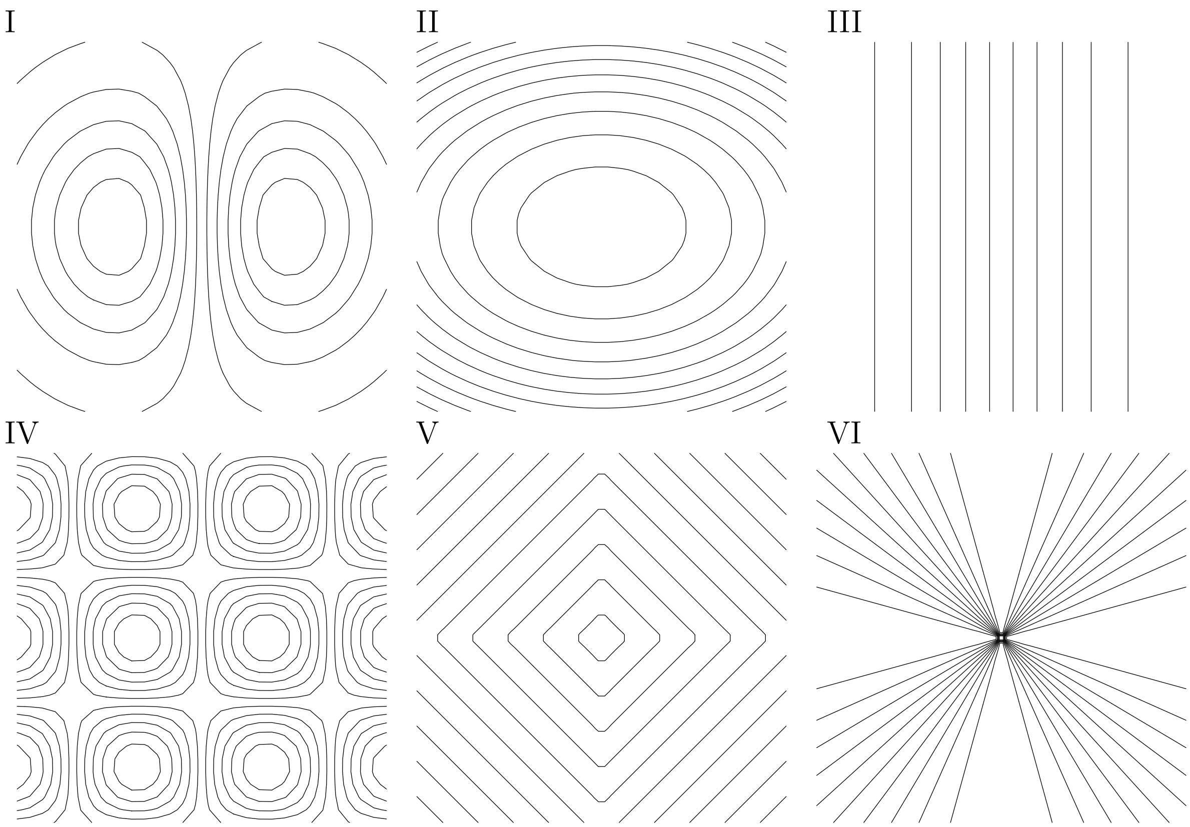
Note. This is seen best in polar coordinates $f(r,\theta) = \cos^2(\theta)\sin^2(\theta)r^4/r^2$ which has an absolute value between 0 and r^2. As $r \to 0$, $f(r,\theta) \to 0$.

T	F	
	*	The scalar projection of a vector $\vec{v}$ onto a vector $\vec{w}$ is always equal to the scalar projection of $\vec{w}$ onto $\vec{v}$.
*		The graph of the function $f(x,y) = \cos(xy)$ can be written as a level surface of a function $g(x,y,z)$.
	*	If the length of the velocity vector $\vec{r}\,'(t)$ does not depend on t then the curvature of the curve $\vec{r}(t)$ is zero.
	*	The curvature of the curve $2\vec{r}(4t)$ at $t=0$ is twice the curvature of the curve $\vec{r}(t)$ at $t=0$.

Note. The curvature of $2\vec{r}(4t)$ at $t=0$ is actually $1/2$ of the curvature of the curve $\vec{r}(t)$ at $t=0$.

Problem 2) (10 points)

Match the contour maps with the corresponding functions $f(x, y)$ of two variables. Note that one of the contour maps is not represented by a formula. No justifications are needed.

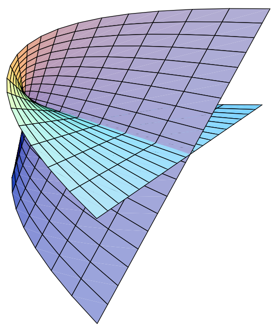


Enter I,II,III,IV,V or VI here	Function $f(x, y)$
III	$f(x, y) = \sin(x)$
II	$f(x, y) = x^2 + 2y^2$
V	$f(x, y) = \|x\| + \|y\|$
I	$f(x, y) = xe^{-x^2-y^2}$
VI	$f(x, y) = x^2/(x^2 + y^2)$

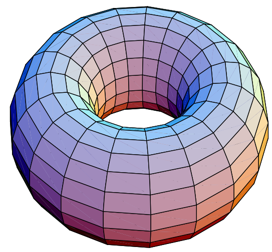
Problem 3) (10 points)

Match the surfaces with their parameterization $\vec{r}(u,v)$ or equation $g(x,y,z)=0$. Note that one of the surfaces is not represented by a formula. No justifications are needed.

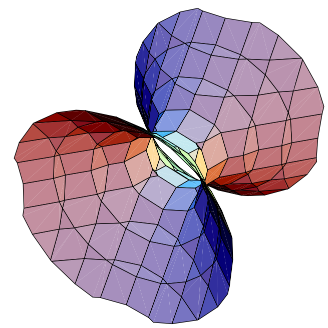
I



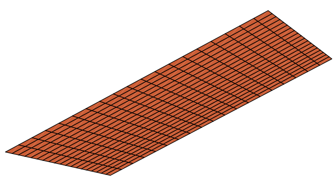
II



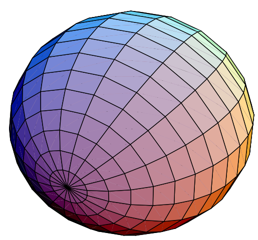
III



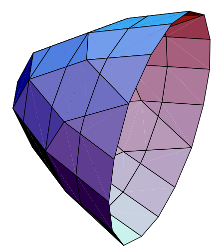
IV



V



VI



Enter I,II,III,IV,V,VI here	Equation or Parameterization
II	$\vec{r}(u,v) = ((1+\sin(u))\cos(v), (1+\sin(u))\sin(v), \cos(u))$
IV	$\vec{r}(u,v) = (v, v-u, u+v)$
I	$\vec{r}(u,v) = (u^2, vu, v)$
III	$x^2 - y^2 + z^2 - 1 = 0$
V	$\vec{r}(u,v) = (\cos(u)\sin(v), \cos(v), \sin(u)\sin(v))$

Problem 4) (10 points)

Find the distance between the two lines

$$\vec{r}_1(t) = (t, 2t, -t)$$

and

$$\vec{r}_2(t) = (1+t, t, t) \ .$$

Solution. $A=(0,0,0)$ is a point on the first curve and $B=(1,0,0)$ is a point on the second curve. The vector $\vec{n} = (1,2,-1)\times(1,1,1) = (3,-2,-1)$ is the direction of the vector connecting the closest points.
The distance is $d = \vec{n}\cdot AB/|\vec{n}| = \boxed{3/\sqrt{14}}$.

Problem 5) (10 points)

a) (6 points) Find a parameterization of the line of intersection of the planes $3x - 2y + z = 7$ and $x + 2y + 3z = -3$.
Hint. Use the fact that the line goes through the point $P = (1, -2, 0)$.

b) (4 points) Find the symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

representing that line.

Solution.
a) The line of intersection has the direction $(3, -2, 1) \times (1, 2, 3) = 8(-1, -1, 1)$. The parameterization is $\boxed{\vec{r}(t) = (1, -2, 0) + t(-1, -1, 1)}$.

b) If a line contains the point (x_0, y_0, z_0) and a vector (a, b, c), then the symmetric equation is $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$. In our case, where $(x_0, y_0, z_0) = (1, -2, 0)$ and $(a, b, c) = (-1, -1, 1)$, the symmetric equations are $\boxed{x - 1 = y + 2 = -z}$.

Problem 6) (10 points)

Find the curvature $\kappa(t)$ of the space curve $\vec{r}(t) = (-\cos(t), \sin(t), -2t)$.

Hint. Use one of the two formulas for the curvature

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} ,$$

where $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$.

Solution. We use the second formula: $\vec{r}\,'(t) = (\sin(t), \cos(t), -2)$. $\vec{r}\,''(t) = (\cos(t), -\sin(t), 0)$. The speed of the curve satisfies $|\vec{r}\,'(t)| = \sqrt{5}$. The vector $\vec{r}\,'(t) \times \vec{r}\,''(t)$ is $(-2\sin(t), -2\cos(t), -1)$ which has length $\sqrt{5}$. therefore, the curvature is constant $\boxed{\kappa(t) = 1/5}$.

Problem 7) (10 points)

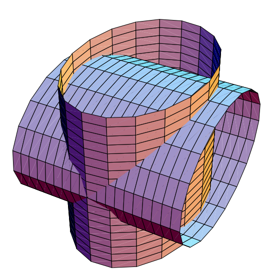
The intersection of the two surfaces $x^2 + \frac{y^2}{2} = 1$ and $z^2 + \frac{y^2}{2} = 1$ consists of two curves.

a) (4 points) Parameterize each curve in the form $\vec{r}(t) = (x(t), y(t), z(t))$.
b) (3 points) Set up the integral for the arc length of one of the curves.
c) (3 points) What is the arc length of this curve?

Solution.
a) Fix first $x(t), y(t)$ to satisfy the first equation then get $z(t) = \cos(t)$ by solving the second equation for z. $\vec{r}(t) = (\cos(t), \sqrt{2}\sin(t), \pm\cos(t))$.

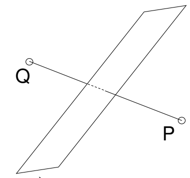
b) $\boxed{\int_0^{2\pi} \sqrt{\cos^2(t) + 2\sin^2(t) + \cos^2(t)}\, dt}$.

c) $\boxed{2\sqrt{2}\pi}$.



Problem 8) (10 points)

Find an equation $ax + by + cz = d$ for the plane which has the property that $Q = (5, 4, 5)$ is the reflection of $P = (1, 2, 3)$ through that plane.



Solution. The plane contains the point $(P+Q)/2 = (6, 6, 8)/2 = (3, 3, 4)$ which is the midpoint between P and Q. The direction of the normal vector to the plane is $\vec{n} = (Q - P) = (4, 2, 2)$. The equation is $4x + 2y + 2z = 12 + 6 + 8 = 26$ or $2x + y + z = 13$.

Problem 9) (10 points)

Let S be the surface given in cylindrical coordinates as $r = 2 + \sin(z)$.

a) (5 points) Find a parameterization

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

of the surface.

b) (5 points) Sketch the surface S. Draw at least three grid curves for each parameter.

Solution. a) The distance to the z-axes is $2 + \sin(z)$. We take the rotation angle θ as a second parameter. Therefore $\vec{r}(\theta, z) = ((2 + \sin(z)) \cos(\theta), (2 + \sin(z)) \sin(\theta), z)$.

b) The grid curves are obtained if one parameter is fixed. If z is fixed, then the curve $\theta \to ((2 + \sin(z)) \cos(\theta), (2 + \sin(z)) \sin(\theta), z)$ defines a circular curve with radius $(2 + \sin(z))$ at height z. If θ is fixed, we obtain curves $z \to ((2 + \sin(z)) \cos(\theta), (2 + \sin(z)) \sin(\theta), z)$.