SOLUTIONS TO 1. PRACTICE EXAM FIRST HOURLY Math 21a, Spring 2003

Problem 1) TF questions (20 points)

Circle for each of the 20 questions the correct letter. No justifications are needed. Your score will be $C-W$ where C is the number of correct answers and W is the number of wrong answers. A (*) indicates the correct solution.

T	*	The vectors $\langle 3,-2,1\rangle$ and $\langle -6,4,2\rangle$ are parallel.
T	*	The length of the vector $\langle 3,4,0\rangle$ is 25.
T	*	For any two vectors, $\vec{v}\times\vec{w}=\vec{w}\times\vec{v}$.
*	F	The vectors $\langle 1,1\rangle$ and $\langle 1,-1\rangle$ are orthogonal.
T	*	For any two vectors $\vec{v},\vec{w}$ one has $\lvert\vec{v}+\vec{w}\rvert^2=\lvert\vec{v}\rvert^2+\lvert\vec{w}\rvert^2$.
*	F	The surface $x^2-y^2+z^2=1$ is a one-sheeted hyperboloid.
*	F	The set of points which have distance 1 from a line is a cylinder.
*	F	If $\lvert\vec{v}\times\vec{w}\rvert=0$ for all vectors $\vec{w}$, then $\vec{v}=\vec{0}$.
T	*	The nonempty intersection of two planes is always a line.
*	F	If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $(\vec{u}\times\vec{v})\times\vec{u}$ is parallel to $\vec{v}$.
T	*	Two nonparallel lines in three dimensional space always intersect in a point.
T	*	Every vector contained in the line $\vec{r}(t)=(1+2t,1+3t,1+4t)$ is parallel to the vector $(1,1,1)$.
*	F	If in spherical coordinates a point is given by $(\rho,\theta,\phi)=(2,\pi/2,\pi/2)$, then its rectangular coordinates are $(x,y,z)=(0,2,0)$.
*	F	If the velocity vector $\vec{r}\,'(t)$ of the planar curve $\vec{r}(t)$ is orthogonal to the vector $\vec{r}(t)$ for all times t, then the curve is a circle.
T	*	Every point on the sphere of radius ρ is determined alone by its angle ϕ from the z axis.
T	*	The equation $r=3$ in cylindrical coordinates is a sphere.
T	*	The set of points which satisfy $x^2-2y^2-3z^2=0$ form an ellipsoid.
*	F	A surface which is given as $r=2+\sin(z)$ in cylindrical coordinates stays the same when we rotate it around the z axis.
T	*	If $\vec{v}\times\vec{w}=\langle 0,0,0\rangle$, then $\vec{v}=\vec{w}$.
T	*	The curvature of a circle of radius r is equal to $1/(2\pi r)$.

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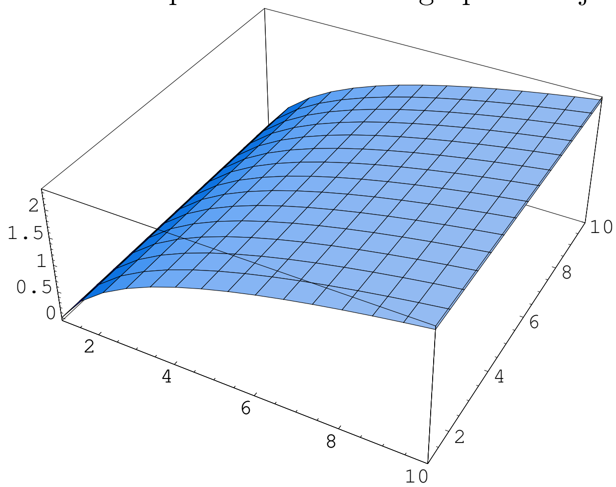
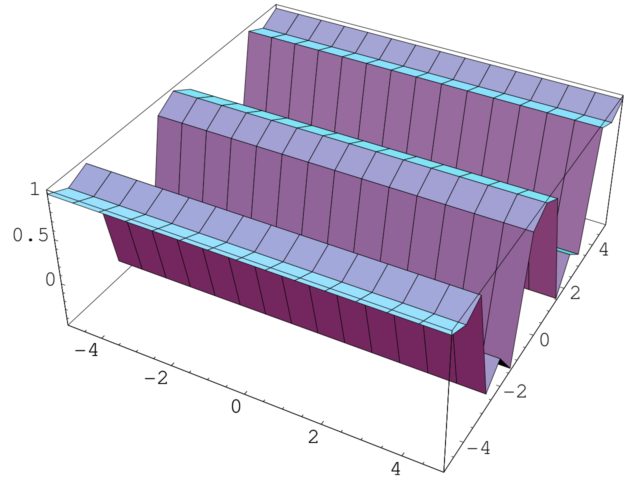
Problem 2) (10 points)

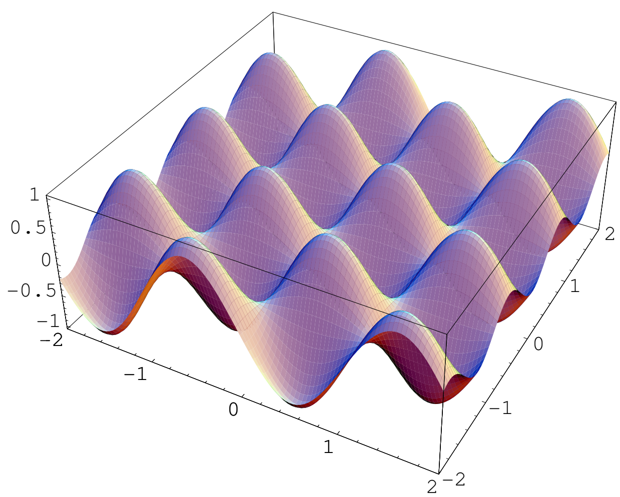
Match the equation with their graphs and justify briefly your choice.



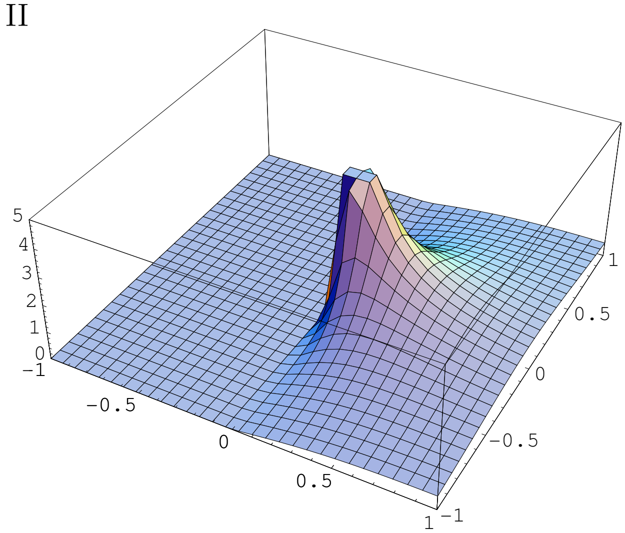
I



II



III

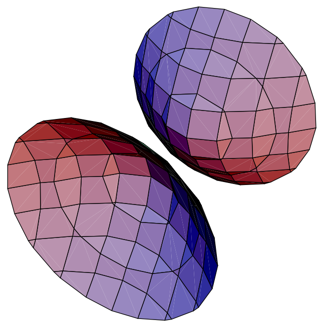


IV

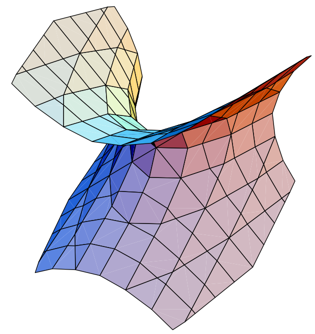
Enter I,II,III,IV here	Equation	Short Justification
III	$z = \sin(3x)\cos(5y)$	two traces show waves
II	$z = \cos(y^2)$	no x dependence, periodic in y
I	$z = \log(x)$	no y dependence, monotone in x
IV	$z = x/(x^2 + y^2)$	singular at (x,y)=(0,0)

Problem 3) (10 points)

Match the equation with their graphs and justify briefly your choice.



I



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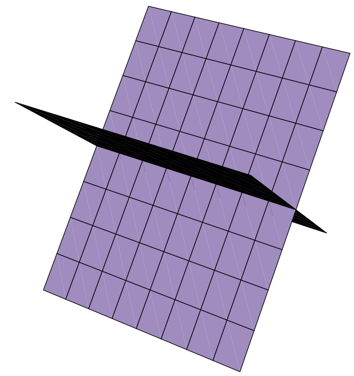

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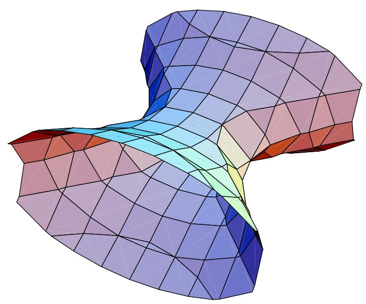
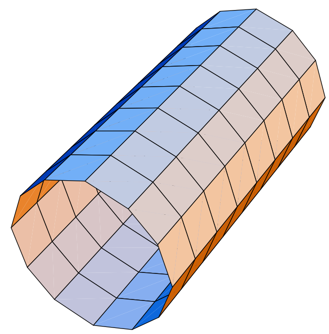
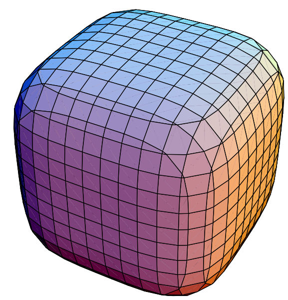


IV



V



VI

Enter I,II,III,IV,V,VI here	Equation	Short explanation
VI	$x^4 + y^4 + z^4 - 1 = 0$	all three traces are circle like
I	$-x^2 + y^2 - z^2 - 1 = 0$	x-y trace is hyperbola, xz trace is empty
V	$x^2 + z^2 = 1$	no y dependence: cylinder
III	$-y^2 + z^2 = 0$	$y^2 = z^2$ means y=z or y=-z, two planes
IV	$x^2 - y^2 + 3z^2 - 1 = 0$	x z trace is ellipse, xy trace is hyperbola
II	$x^2 - y - z^2 = 0$	parabolas and hyperbola appear as traces

Problem 4) (10 points)

Given the vectors $v = \langle 1,1,0\rangle$ and $w = \langle 0,0,1\rangle$ and the point $P = (2,4,-2)$. Let Σ be the plane which goes through the origin and contains the vectors v and w.
a) Determine the distance from P to the origin.

Solution: $\sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$.

b) Determine the distance from P to the plane Σ.

Solution: $\Sigma : x - y = 0$, $n = \langle 1,-1,0\rangle$. $Q = (0,0,0)$ is a point on the plane. $\vec{PQ} \cdot n/|n| = \langle 2,4,-2\rangle \cdot \langle 1,-1,0\rangle/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$

Problem 5) (10 points)

Boba Fett is flying through the air when his rocket pack malfunctions and sends him spinning out of control. At time $t = 0$, he is at the point $P_0 = (0,0,27)$ and moving with velocity $\vec{v} = \langle 10,0,0\rangle$. While he is in the air, his acceleration is given by $\vec{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t + 2t, -6t\rangle$ for $t \geq 0$.

1. For $t \geq 0$, find Boba's position as a function of time.

   Solution: Integrate $r''(t)$ and compare $r'(0) = (10,0,0)$, then integrate again and compare with $\vec{r}(0) = \langle 0,0,27\rangle$. Result: $\vec{r}(t) = \langle 10t + \pi t - \sin(\pi t), -\cos(\pi t) + t^3/3 + 1, 27 - t^3\rangle$.

2. The ground is represented by the xy plane. At what time does Boba hit the ground? What are the x and y coordinates of the point, where he hits?

   Solution: $27 - t^3 = 0$ means $t = 3$. At that time $\vec{r}(3) = (30 + 3\pi, 11, 0)$.

Problem 6) (10 points)

a) Calculate the unit tangent vector T, the unit normal vector N as well as the binormal vector B for the curve $\vec{r}(t) = \langle \cos(t), \sqrt{3}t, \sin(t)\rangle$ at the point $t = \pi$.

Solution:
$\vec{T}(t) = \langle -\sin(t)/2, \sqrt{3}/2, \cos(t)/2\rangle$.
$\vec{N}(t) = \langle -\cos(t), 0, \sin(t)\rangle$.
$\vec{B}(t) = \langle -\sqrt{3}\sin(t)/2, -1/2, \sqrt{3}\cos(t)/2\rangle$.

At the time $t = \pi$, we have
$\vec{T}(\pi) = \langle 0, \sqrt{3}/2, -1/2\rangle$.
$\vec{N}(\pi) = \langle 1,0,0\rangle$.
$\vec{B}(\pi) = \langle 0, -1/2, -\sqrt{3}/2\rangle$.

b) Verify that for a general curve the formula $\frac{d}{dt}\vec{B}(t) = \vec{T}(t) \times \vec{N}'(t)$ holds.

Solution:

$$\frac{d}{dt}\vec{B}(t) = \frac{d}{dt}\vec{T}(t) \times \vec{N}(t) = \vec{T}'(t) \times \vec{N}(t) + \vec{T}(t) \times \vec{N}'(t) = 0 + \vec{T}(t) \times \vec{N}'(t) \ .$$

We have used that $\vec{T}'(t) \times \vec{N}(t) = \vec{T}'(t) \times \vec{T}'(t)/|T'(t)| = \vec{0}$.

Hint. Here are the formulas for the unit tangent vector, the unit normal vector as well as the binormal vector.

$$\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$$
$$\vec{N}(t) = \vec{T}'(t)/|T'(t)|$$
$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t).$$

Problem 7) (10 points)

a) Identify the surface whose equation is given in spherical coordinates as $\theta = \pi/4$.

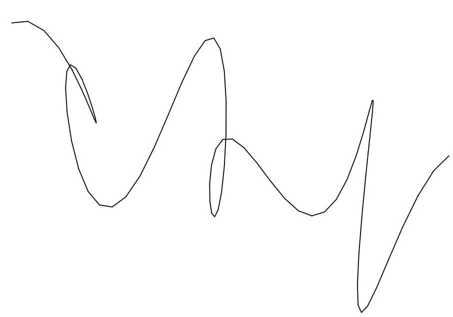
Solution: A half plane contained in $x = y$.

b) Identify the surface whose equation is given in spherical coordinates as $\phi = \pi/4$.
Solution: A half cone. $x^2 + y^2 = z^2$ with $z \geq 0$.

c) Identify the surface, whose equation is given in cylindrical coordinates by $z = r^2$. Either name it or sketch the surface convincingly.

Solution: $z = x^2 + y^2$ is a circular paraboloid.

Problem 8) (10 points)



Let $\vec{r}(t)$ be the space curve $\vec{r}(t) = (t^2, \sin(3\pi t), \cos(5\pi t))$.

a) Calculate the velocity, the acceleration and the speed of $\vec{r}(t)$ at time $t = 1$.

Solution:
$\vec{v}(t) = \vec{r}'(t) = (2t, 3\pi\cos(3\pi t), -5\pi\sin(5\pi t))$.
$\vec{v}(1) = (2, -3\pi, 0)$

$\vec{a}(t) = \vec{r}''(t) = (2, -9\pi^2 \sin(3\pi t), -25\pi^2 \cos(5\pi t))$.
$\vec{a}(1) = (2, 0, 25\pi^2)$.
$|v(1)| = \sqrt{4 + 9\pi^2}$.

b) Write down the length of the curve from $t = 1$ to $t = 10$ as integral. You don't have to evaluate the integral.

Solution:
$\int_1^1 0\sqrt{4t^2 + 9\pi^2 \cos^2(3\pi t) + 25\pi^2 \sin^2(5\pi t)}$.

c) The curve $t \mapsto \vec{r}(t) = (t^3, 1 - t, 1 - t^3)$ lies in a plane. What is the equation of this plane?

Solution: $t = 0 : P = (0, 1, 1), t = 1 : Q = (1, 0, 0), t = 2 : R = (8, -1, -7)$ are points on the Plane. $\vec{PQ} = (1, -1, -1)$, $\vec{PR} = (8, -2, -8)$. Their cross product is $(6, 0, 6)$. The plane is $x + z = 1$.

Problem 9) (10 points)

Let S be the surface given by

$$z^2 = \frac{x^2}{4} + y^2 \ .$$

a) Sketch the surface S.

Solution: It is an elliptical cone. Every z-trace is an ellipse.
b) Let (a, b, c) be a point on the surface S. Find a parametric equation for the line that passes through (a, b, c) and lies entirely on the surface S.

Solution: The line $\vec{r}(t) = (at, bt, ct)$ lies on the surface.