2/5/2003, VECTORS/DOT PRODUCT — O. Knill, Math 21a

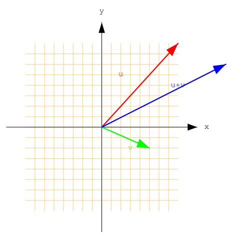
HOMEWORK FOR FRIDAY: Section 9.2: 20, 34, Section 9.3: 18, 34, 38

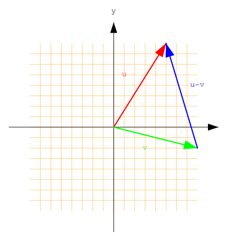
VECTORS. Two points $P_1 = (x_1, y_1, z_1)$, $Q = P_2 = (x_2, y_2, z_2)$ determine a **vector** $\vec{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. It points from P_1 to P_2 and we can write $P_1 + \vec{v} = P_2$.

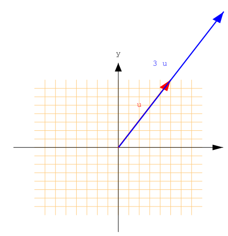
COORDINATES. Points P in space are in one to one correspondence to vectors pointing from 0 to P. The numbers $\vec{v}_i$ in a vector $\vec{v} = (v_1, v_2, v_3)$ are also called **coordinates** of the vector.

REMARK: vectors can be drawn **everywhere** in space. If a vector starts at 0, then the vector $\vec{v} = (v_1, v_2, v_3)$ points to the point (v_1, v_2, v_3). That's is why one can identify points $P = (a, b, c)$ in space with a vector $\vec{v} = (a, b, c)$. Two vectors which are translates of each other are considered **equal**.

ADDITION SUBTRACTION, SCALAR MULTIPLICATION.





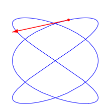


BASIS VECTORS. The vectors $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$ are called **basis vectors**.

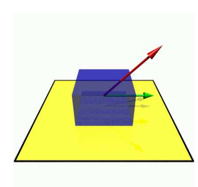
Every vector $\vec{v} = (v_1, v_2, v_3)$ can be written as a sum of basis vectors: $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$.

WHERE DO VECTORS OCCUR?

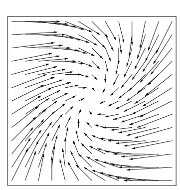
Velocity (see later): if $(f(t), g(t))$ is a point in the plane which depends on time t, then $\vec{v} = (f'(t), g'(t))$ is the **velocity vector** at the point $(f(t), g(t))$.



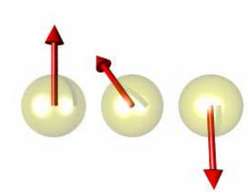
Forces: static problems involve the determination of a force on objects. Vectors appear also when describing fields like the electric field or a wind velocity field.



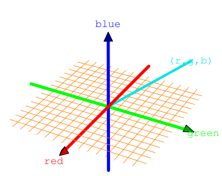
Fields: fields like electromagnetic or gravitational fields or velocity fields in fluids are described with vectors.

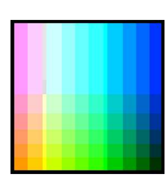


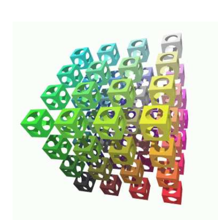
Qbits: in quantum computation, one does not work with bits, but with bits, which are vectors.



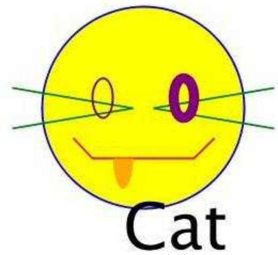
Color Any color can be written as a vector $\vec{v} = (r, g, b)$, where r is the red component, g is the green component and b is the blue component.







Svg or Flash. Scalable Vector Graphics is an emerging standard for the web. It might rival soon Flash which is currently the most popular vector based animation tool. From www.w3.org: "SVG is a language for describing two-dimensional graphics in XML. SVG allows for three types of graphic objects: vector graphic shapes (e.g., paths consisting of straight lines and curves), images and text. Graphical objects can be grouped, styled, transformed and composited into previously rendered objects.



VECTOR OPERATIONS: The addition and scalar multiplication of vectors satisfy "obvious" properties: (no need memorizing them). We write $*$ for multiplication with a scalar.

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$	commutativity
$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$	additive associativity
$\vec{u} + 0 = \vec{u} + 0$	null vector
$r * (s * \vec{v}) = (r * s) * \vec{v}$	scalar associativity
$(r + s)\vec{v} = \vec{v}(r + s)$	distributivity in scalar
$r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$	distributivity in vector
$1 * \vec{v} = \vec{v}$	one element

LENGTH. The length $||\vec{v}||$ of $\vec{v}$ is the distance from the beginning to the end of the vector.

EXAMPLEs. If $\vec{v} = (3, 4, 5)$, then $||\vec{v}|| = \sqrt{50} = 5\sqrt{2}$. TRIANGLE INEQUALITY: $||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$.

UNIT VECTOR. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/||\vec{v}||$ is a unit vector.
EXAMPLE: If $\vec{v} = (3, 4)$, then $\vec{v} = (2/5, 3/5)$ is a unit vector, $\vec{i}, \vec{j}, \vec{k}$ are unit vectors.

PARALLEL VECTORS. Two vectors $\vec{v}$ and $\vec{w}$ are called **parallel**, if $\vec{v} = r\vec{w}$ with some constant r.

DOT PRODUCT. The **dot product** of two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is defined as $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$. Other notations are $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$ or $< \vec{v}|\vec{w} >$ (quantum mechanics) or $v_i w^i$ (Einstein notation) or $g_{ij} v^i w^j$ (general relativity). The dot product is also called **scalar product**, or **inner product**.

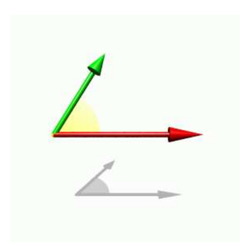
LENGTH. Using the dot product one can express the length of $\vec{v}$ as $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$.

CHALLENGE. Express the dot product in terms of length only!

SOLUTION: $(\vec{v} + \vec{w}, \vec{v} + \vec{w}) = (\vec{v}, \vec{v}) + (\vec{w}, \vec{w}) + 2(\vec{v}, \vec{w})$ can be solved for $(\vec{v}, \vec{w})$.

ANGLE. Because $||\vec{v} - \vec{w}||^2 = (\vec{v} - \vec{w}, \vec{v} - \vec{w}) = ||\vec{v}||^2 + ||\vec{w}||^2 - 2(\vec{v}, \vec{w})$ is by the **cos-theorem** equal to $||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}|| \cdot ||\vec{w}|| \cos(\phi)$, where ϕ is the angle between the vectors $\vec{v}$ and $\vec{w}$, we have the important formula

$$\boxed{\vec{v} \cdot \vec{w} = ||\vec{v}|| \cdot ||\vec{w}|| \cos(\phi)}$$



FINDING ANGLES BETWEEN VECTORS. Find the angle between the vectors $(1, 4, 3)$ and $(-1, 2, 3)$.
ANSWER: $\cos(\phi) = 16/(\sqrt{26}\sqrt{14}) \sim 0.839$. So that $\phi = \arccos(0.839..) \sim 33°$.

ORTHOGONALITY. Two vectors are called **orthogonal** if $v \cdot w = 0$. The zero vector $\vec{0}$ is orthogonal to any vector.

PROJECTION. The vector $\boxed{\vec{a} = \vec{w}(\vec{v} \cdot \vec{w}/|\vec{w}|^2)}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** is defined as $\boxed{(\vec{v} \cdot \vec{w}/||\vec{w}||)}$. (Its absolute value is the length of the projection.)
The vector $\vec{b} = \vec{v} - \vec{a}$ is the **component** of $\vec{v}$ orthogonal to the $\vec{w}$-direction.