Homework for Monday 3/17/2003:
Section 11.6: 20, 26, 30, 42, 44

DEFINITIONS.
$\nabla f(x,y) = (f_x, f_y)$, $\nabla f(x,y,z) = (f_x, f_y, f_z)$, **gradient** ∇ is spelled "Nabla". (=name of an Egyptian harp, the Hebrew word "nevel"=harp seems to have the same aramaic origin).
$D_{\vec{v}} f = \nabla f \cdot \vec{v}$ **directional derivative**

FACTS:
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$ **chain rule**.
$\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x,y,z) = c$ where $c = f(x_0, y_0, z_0)$.
$\frac{d}{dt} f(\vec{x} + t\vec{v}) = D_{\vec{v}} f$ by chain rule.
$\frac{x-x_0}{f_x(x_0,y_0,z_0)} = \frac{y-y_0}{f_y(x_0,y_0,z_0)} = \frac{z-z_0}{f_z(x_0,y_0,z_0)}$ **normal line** to $f(x,y,z) = c$ at (x_0, y_0, z_0).
$(x-x_0) f_x(x_0,y_0,z_0) + (y-y_0) f_y(x_0,y_0,z_0) + (z-z_0) f_z(x_0,y_0,z_0) = 0$ **tangent plane** at (x_0, y_0, z_0)
The **directional derivative** is maximal $|\nabla f|^2$ in the $\vec{v} = \nabla f$ direction.
$f(x,y)$ **increases** if we walk on the xy-plane in the ∇f direction.
Partial derivatives are special directional derivatives.
If $D_{\vec{v}} f(\vec{x}) = 0$ for all $\vec{v}$, then $\nabla f(\vec{x}) = \vec{0}$.

ALGORITHMS.
Implicit differentiation: $f(x, y(x)) = 0$, compute y_x without knowing $y(x)$: $f_x 1 + f_y y'(x) = 0$ gives $y'(x) = -f_x/f_y$. Compute directional derivatives. Find direction where directional derivative is maximal or minimal.

EXAMPLE PROBLEM. Show that the two surfaces $x^2 + y^2 = z^2, x^2 + y^2 + z^2 = 1$ intersect at a right angle at all points of intersection.

SOLUTION. The angle of intersection is the angle between the normal vectors which are $(2x, 2y, -2z)$ and $(2x, 2y, 2z)$. The dot product is $4x^2 + 4y^2 - 4z^2 = 0$ (using the first equation).

WHERE DOES THE CHAIN RULE MATTER (informal)

- The chain rule is used in **change of variable** formulas. For example, if f is a function in the plane and $x = r\cos(\theta), y = r\sin(\theta)$ is a change of coordinates into polar coordinates, then calculating the gradient of f in the new coordinates requires the chain rule.

- The chain rule will be used in the **fundamental theorem for line integrals**: $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(b) - f(a)$ which appears later in this course. The chain rule said that inside the integral, we have $d/dt f(\vec{r}(t))$.

- **Gradients are orthogonal to level surfaces**: assume we have a curve $\vec{r}(t)$ on a surface $g(x,y,z) = c$. Because we move on the surface where g is constant, we have $d/dt g(\vec{r}(t)) = 0$. The chain rule says $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$. In other words, the gradient of g is orthogonal to the surface $g = const$. (We had argued earlier using the linear approximation of g.)

- The chain rule illustrates also the **Lagrange multiplier** method which we will see later. If we want to extremize $f(x,y)$ on the constraint $g(x,y)$ and $\vec{r}(t)$ is a curve on $g(x,y) = c$, then at a critical point, we must have $d/dt f(\vec{r}(t)) = 0$. With the chain rule one can see that the velocity vector $\vec{r}\,'(t)$ is orthogonal to $\nabla f(\vec{r}(t))$. Because the velocity vector is orthogonal to ∇g The vectors ∇g and ∇f have to be parallel.

- In **fluid dynamics**, you often see partial differential equations starting with $u_t + u\nabla u$, where u is the velocity of the fluid. For example, the conservation of momentum of an ideal fluid in the presence of an external force f is $u_t + u \cdot \nabla u = (1/\rho)\nabla p + f$, where p is the pressure and ρ is the density. What does this mean? The term $u_t + u \cdot \nabla u$ is the change of velocity u in the coordinate frame of a particle in the fluid. ($u(t,x,y,z) = (u_1(t,x,y,z), u_2(x,y,z), u_3(x,y,z)$ is a vector and ∇u means applying the gradient to each coordinate). For any quantity f like pressure, vorticity, density etc. , $Df/Dt = f_t + u \cdot \nabla f$ means the time derivative of f **moving with the fluid**.

- In **chaos theory**, one wants to understand what happens after iterating a map f. If $f^{(n)}(x) = f f^{(n-1)}(x)$, then $(f^{(n)})'(x) = f'(f^{(n-1)}x) f'(f^{(n-2)}(x)) ... f'(x)$ is a product of matrices. Chaos means that $(f^n)'(x)$ grows exponentially for a large set of x. A measure of chaos is the Lyapunov exponent $\lambda = n^{-1} \log|(f^n)'(x)|$ in the limit $n \to \infty$. If this number is positive, one has sensitive dependence on initial conditions. The map $x \mapsto 4x(1-x)$ on the interval $[0,1]$ for example has this property. Small chaos of order $\lambda = 10^{-15}$ is present in the solar system. Large chaos $\lambda = 10^{15}$ is present in a small volume of argon gas at room temperature.

Let $f(x, y, z) = z + x + \sin(z)$ and $\vec{r}(t) = (\cos(t), \sin(t), t)$.
Find $\frac{d}{dt} f(r(t))$ at the point $t = 0$ by differentiating the function $t \mapsto f(\vec{r}(t))$.

Find $\frac{d}{dt} f(\vec{r}(t))$ at the point $t = 0$ using the chain rule.

The equation $f(x, y, z) = 0$ defines $z = z(x, y)$. Find $\partial z(x, y)/\partial x$ at the point $(0, 0)$.