4/8/2003, CYLINDRICAL/SPHERICAL COORDINATES	Math 21a, O. Knill

HOMEWORK. Section 12.8: 4, 8, 10, 20, 32

REMINDER: INTEGRATION POLAR COORDINATES.

$$\int\int_R f(r,\theta)\,\boxed{r}\,d\theta dr\ .$$

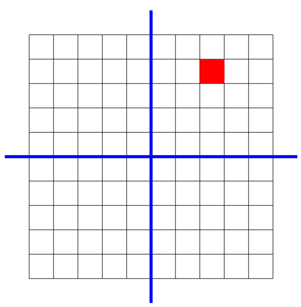
EXAMPLE 1. Area of a disk of radius R

$$\int_0^R \int_0^{2\pi} r\ d\theta dr = 2\pi \frac{r^2}{2}|_0^R = R^2\pi\ .$$

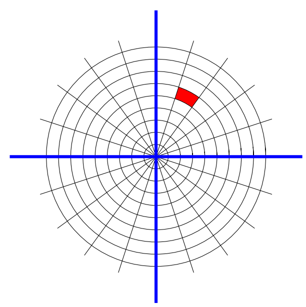
WHERE DOES THE FACTOR "r" COME FROM?

1. EXPLANATION. A small rectangle with dimensions $d\theta dr$ in the (r,θ) plane is mapped to a sector segment in the (x,y) plane. It has approximately the area $rd\theta dr$. It is small for small r.

2. EXPLANATION (more details on Friday). Look at the map $(r,\theta) \mapsto (r\cos(\theta, r\sin(\theta))(f(r,\theta), g(r,\theta))$ which changes from Cartesian coordinates to polar coordinates. The **Jacobean** is $T' = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{bmatrix} = \begin{bmatrix} f_r & f_\theta \\ g_r & g_\theta \end{bmatrix}$ with determinant $f_r g_\theta - f_\theta g_r = r$. This is a special case of a more general formula.

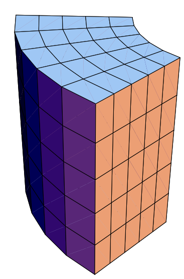


$$T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$$



CYLINDRICAL COORDINATES. Use polar coordinates in the x-y plane and leave the z coordinate. Take $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$. The integration factor r is the same as in polar coordinates.

$$\int\int\int_{T(R)} f(x,y,z)\ dxdydz = \int\int\int_R f(r,\theta,z)\,\boxed{r}\,drd\theta dz$$



COORDINATES OF CAMBRIDGE. On the website $http://cello.cs.uiuc.edu/cgi-bin/slamm/ip2ll/$ you can enter a host like $www.math.harvard.edu$ and get latitude and longitude of the host: $(lat, lon) = (42.365, -71.1)$. Using (r,θ,ϕ) coordinates, we obtain the position $(r, 90-42.365, -71.1)$ of the host in spherical coordinates. The site does not give the height, but we are about on see-level, so that $r = 6365km$.

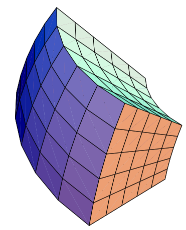
EXAMPLE. Calculate the volume bounded by the parabolic $z = 1-(x^2+y^2)$ and the $x-y$ plane. In cylindrical coordinates, the paraboloid is $z(r,\phi) = 1-r^2$:

$$\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} rdzd\phi dr = \int_0^1 \int_0^{2\pi} (r-r^3)\ d\phi dr = 2\pi(r^2/2 - r^4/4)_0^1 = \pi\ .$$

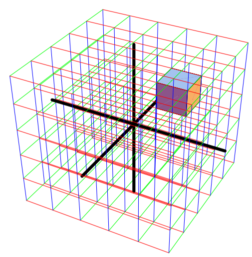
SPHERICAL COORDINATES. Spherical coordinates use the radius ρ as well as two angles: θ the polar angle and ϕ, the angle between the vector and the z axes. The coordinate change is

$$T : (x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) .$$

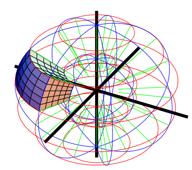
The integration factor can be seen from the dimensions of a spherical wedge with dimensions $d\rho, \rho\sin(\phi)\, d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.



$$\int\int\int_{T(R)} f(x, y, z)\; dxdydz = \int\int\int_{R} f(\rho, \theta, z)\; \boxed{\rho^2\sin(\phi)}\; d\rho d\theta d\phi$$



$$T : (\rho, \theta, \phi) \mapsto (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$$



VOLUME OF SPHERE. A sphere of radius R has the volume

$$\int_0^R\int_0^{2\pi}\int_0^{\pi} \rho^2\sin(\phi)\; d\phi d\theta d\rho .$$

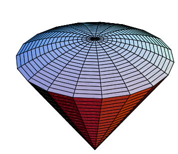
The most inner integral $\int_0^\pi \rho^2\sin(\phi)d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\; d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\; d\rho = 4\pi R^3/3$.

MOMENT OF INERTIA. The moment of inertia of a body G with respect to an axes L is the triple integral $\int\int\int_G r(x, y, z)^2\; dzdydx$, where $r(x, y, z) = R\sin(\phi)$ is the distance from L. Problem: calculate the moment of inertia of a sphere of radius R with respect to the z-axes:

$$I = \int_0^R\int_0^{2\pi}\int_0^{\pi} \rho^2\sin^2(\phi)\rho^2\sin(\phi)\; d\phi d\theta d\rho = (\frac{1}{3}\sin^3(\phi)_0^\pi)(\int_0^R \rho^4\; dr)(\int_0^{2pi}\; d\theta) = \frac{4}{3}\cdot\frac{R^5}{5}\cdot 2\pi = \frac{8\pi R^5}{15} = \frac{VR^2}{5} .$$

If a sphere spins around the z-axes with angular velocity ω, then $I\omega^2/2$ is the kinetic energy of that sphere. Example: the moment of inertia of the earth is $810^{37} kgm^2$. The angular velocity is $\omega = 1/day = 1/(86400s)$ so that the energy of the earth rotation is $810^{37} kgm^2/(7464960000s^2) \sim 10^{28}J = 10^{25}kJ \sim 2.510^{24}kcal$.

DIAMOND. Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.



Solution: we use spherical coordinates to find the center of mass $(\overline{x}, \overline{y}, \overline{z})$:

$V = \int_0^1\int_0^{2\pi}\int_0^{\pi/6} \rho^2\sin(\phi)\; d\phi d\theta d\rho = \frac{(1-\frac{\sqrt{3}}{2})}{3} 2\pi$

$\overline{x} = \int_0^1\int_0^{2\pi}\int_0^{\pi/6} \rho^3\sin^2(\phi)\cos(\theta)\; d\phi d\theta d\rho/V = 0$

$\overline{y} = \int_0^1\int_0^{2\pi}\int_0^{\pi/6} \rho^3\sin^2(\phi)\sin(\theta)\; d\phi d\theta d\rho/V = 0$

$\overline{z} = \int_0^1\int_0^{2\pi}\int_0^{\pi/6} \rho^3\cos(\phi)\sin(\phi)\; d\phi d\theta d\rho/V = \frac{2\pi}{32V}$