

GENERAL TIPS.

- Do at least one practice exams, online TF questions.
- Make list of facts on a sheet of paper.
- Fresh up short-term memory before test.
- Review homework. Find error patterns.
- During the exam: read the questions carefully. Wrong understanding could lead you to solve an other problem:

• Ask questions:

"Ask a question and you're a fool for three minutes; do not ask a question and you're a fool for the rest of your life." - Chinese Proverb

There was a college student trying to earn some pocket money by going from house to house offering to do odd jobs. He explained this to a man who answered one door. "How much will you charge to paint my porch?" asked the man. "Forty dollars." "Fine" said the man, and gave the student the paint and brushes. Three hours later the paint-splattered lad knocked on the door again. "All done!", he says, and collects his money. "By the way," the student says, "That's not a Porsche, it's a Ferrari."

MIDTERM TOPICS.

- Properties of dot, cross and triple product
- Orthogonal, parallel, projection
- Parametrized Lines and Planes
- Switch between parameterization and equations
- Given line and plane, find intersection
- Given plane and plane, find intersection
- Given line and point, find plane
- Given point and point, find line
- Given three points, find plane
- Distances: point-line, line-line, point-plane
- Distinguish and analyse curves
- Determine curves from acceleration
- Calculate curvature, \vec{T} , \vec{N} , \vec{B} .
- Continuity of functions $f(x, y)$.
- Distance between two lines
- Distance between two planes
- Angle between two vectors
- Angle between two planes
- Area of parallelogram, triangle in space
- Volume of parallelepiped
- Identify surfaces in spherical coordinates
- Identify surfaces in cylindrical coordinates
- Distinguish parametric surfaces, contour maps, quadrics, graphs
- Traces, intercepts, generalized traces, grid curves
- Compute velocity, acceleration
- Find length of curves
- Parameterize curves by intersecting two surfaces
- Level=contour curves, level=contour surfaces

VECTORS.

Two points $P = (1, 2, 3)$, $Q = (3, 4, 6)$ define a vector $\vec{v} = \vec{PQ} = \langle 2, 2, 3 \rangle$. If $\vec{v} = \lambda \vec{w}$, then the vectors are **parallel** if $\vec{v} \cdot \vec{w} = 0$, then the vectors are called **orthogonal**. For example, $(1, 2, 3)$ is parallel to $(-2, -4, -6)$ and orthogonal to $(3, -2, 1)$. The addition, subtraction and scalar multiplication of vectors is done componentwise. For example: $(3, 2, 1) - 2((1, 1, 1) + (-1, -1, 0)) = (3, 2, -1)$.

A nonzero vector \vec{v} and a point $P = (x_0, y_0, z_0)$ define a line $\vec{r}(t) = P + t\vec{v}$. Two nonzero, nonparallel vectors \vec{v}, \vec{w} and a point P define a plane $P + t\vec{v} + s\vec{w}$. The vector $\vec{n} = \vec{v} \times \vec{w} = (a, b, c)$ is orthogonal to the plane. Points on the line satisfy the symmetric equation $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. Points on the plane satisfy an equation $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$. Using the dot product for projection and the vector product to get orthogonal vectors, one can solve many geometric problems in 3D.

DOT PRODUCT (is scalar)

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

$$|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$$

$$(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$$

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

$$\{1, 2, 3\} \cdot \{3, 4, 5\}$$

$$\frac{d}{dt}(\vec{v} \cdot \vec{w}) = \left(\frac{d}{dt}\vec{v}\right) \cdot \vec{w} + \vec{v} \cdot \left(\frac{d}{dt}\vec{w}\right)$$

commutative

angle

linearity

distributivity

in Mathematica

product rule

CROSS PRODUCT (is vector)

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$$

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$$

$$(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})$$

$$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

$$\text{Cross}[\{1, 2, 3\}, \{3, 4, 5\}]$$

$$\frac{d}{dt}(\vec{v} \times \vec{w}) = \left(\frac{d}{dt}\vec{v}\right) \times \vec{w} + \vec{v} \times \left(\frac{d}{dt}\vec{w}\right)$$

anti-commutative

angle

linearity

distributivity

Mathematica

product rule

PROJECTIONS.

Projection:

$$\text{proj}_{\vec{v}}(\vec{w}) = \frac{(\vec{v} \cdot \vec{w})\vec{v}}{|\vec{v}|^2}.$$

Is a vector parallel to \vec{v} .

Scalar projection:

$$\text{comp}_{\vec{v}}(\vec{w}) = |\text{proj}_{\vec{v}}(\vec{w})| = \frac{|\vec{v} \cdot \vec{w}|}{|\vec{v}|}$$

the length of the projected vector.

Applications:

- Distance $P+t\vec{v}, Q+s\vec{w}$ is scalar projection of \vec{PQ} onto $\vec{v} \times \vec{w}$.
- Distance $P, Q+t\vec{v}+s\vec{w}$ is scalar projection of \vec{PQ} onto $\vec{n} = \vec{v} \times \vec{w}$.

SURFACES.

$\{g(x, y, z) = C\}$ define in general surfaces. Examples are **graphs**, where $g(x, y, z) = z - f(x, y) = 0$ or planes, where $g(x, y, z) = ax + by + cz = C$. If g has quadratic or linear terms only, the surface is called a **quadric**: example $x^2 + xy + y^2 = -z^2 + 2x = 0$. Some surfaces are sometimes easier to describe in cylindrical or spherical coordinates: example sphere: $\rho = \text{const}$ or cylinder: $r = \text{const}$.

Surfaces can be analyzed by looking at **traces**, intersections with planes parallel to the coordinate planes. This is especially true for graphs, where the traces $f(x, y) = C$ are called **contour lines**. Examples are isobars, isotherms or topographical contour lines.

QUADRICS CHECKLIST. Quadrics like:

- ellipsoid, sphere
- cylinder
- hyperbolic cylinder
- cone
- one sheeted hyperboloid
- two sheeted hyperboloid
- paraboloid
- hyperbolic paraboloid

can be identified using **traces**, the intersections with planes.

CURVES.

$\vec{r}(t) = (x(t), y(t), z(t)), t \in [a, b]$ defines a curve. By differentiation, we obtain the **velocity** $\vec{r}'(t)$ and **acceleration** $\vec{r}''(t)$. If we integrate the speed $|\vec{r}'(t)|$ over the interval $[a, b]$, we obtain the **length** of the curve.

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

Example: $\vec{r}(t) = (1, 3t^2, t^3), \vec{r}'(t) = (0, 6t, 3t^2)$, so that $|\vec{r}'(t)| = 3t(4 + t^2)$. The length of the curve between 0 and 1 is $\int_0^1 3t(4 + t^2) dt = 6t^2 + 3\frac{t^4}{4} \Big|_0^1 = 6 \cdot \frac{3}{4}$.

CURVATURE, $\vec{T}, \vec{N}, \vec{B}$

$\vec{r}(t) = (x(t), y(t), z(t))$ **curve**
 $\vec{r}'(t)$ **velocity**
 $\vec{r}''(t)$ **acceleration**

$\vec{T} = \vec{r}'(t)/|\vec{r}'(t)|$ **unit tangent vector**
 $\vec{N} = \vec{T}'(t)/|\vec{T}'(t)|$ **unit normal vector**
 $\vec{B} = \vec{T} \times \vec{N}$ **binormal vector**
 $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$ **curvature**

COORDINATE SYSTEMS.

rectangular	cylindrical	spherical
(x, y, z)	(r, θ, z)	(ρ, θ, ϕ)
x real	$r \geq 0$	$\rho \geq 0$
y real	$\theta \in [0, 2\pi)$	$\theta \in [0, 2\pi)$
z real	z real	$\phi \in [0, \pi]$

$$\begin{aligned} x &= r \cos(\theta) \\ y &= r \sin(\theta) \\ z &= z \end{aligned}$$

$$\begin{aligned} x &= \rho \cos(\theta) \sin(\phi) \\ y &= \rho \sin(\theta) \sin(\phi) \\ z &= \rho \cos(\phi) \end{aligned}$$