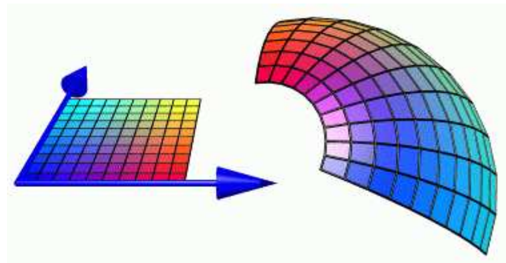
Homework: Section 12.3: 30, 36 — Section 12.6: 8, 10, 22

SURFACE AREA

$$\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv$$

is the area of the surface.

INTEGRAL OF A SCALAR FUNCTION ON A SURFACE. If S is a surface, then $\int\int_S f(x,y)\, dS$ should be an average of f on the surface. If $f(x,y) = 1$, then $\int\int_S \, dS$ should be the area of the surface. If S is the image of $\vec{r}$ under the map $(u,v) \mapsto \vec{r}(u,v)$, then $dS = |\vec{r}_u \times \vec{r}_v| \, dudv$.



DEFINITION. Given a surface $S = \vec{r}(R)$, where R is a domain in the plane and where $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$. The surface integral of $f(u,v)$ on S is defined as

$$\int\int_S f \, dS = \int\int_R f(u,v) |\vec{r}_u \times \vec{r}_v| \, dudv \, .$$

INTERPRETATION. If $f(x,y)$ measures a quantity then $\int\int_S f \, dS$ is the average of the function f on S.

EXPLANATION OF $|\vec{r}_u \times \vec{r}_v|$. The vector $\vec{r}_u$ is a tangent vector to the curve $u \mapsto \vec{r}(u,v)$, when v is fixed and the vector $\vec{r}_v$ is a tangent vector to the curve $v \mapsto \vec{r}(u,v)$, when u is fixed. The two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A little rectangle spanned by $[u, u+du]$ and $[v, v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r} + \vec{r}_u]$ and $[\vec{r}, \vec{r} + \vec{r}_v]$.

A simple case: consider $\vec{r}(u,v) = (2u, 3v, 0)$. This surface is part of the xy-plane. The parameter region R just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = (0,0,6)$ and we see for example that the area of $\vec{r}(R)$ is 6 times the area of R.

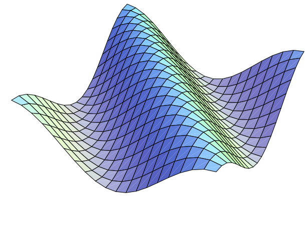
POLAR COORDINATES. If we take $\vec{r}(u,v) = (u\cos(v), u\sin(v), 0)$, then the rectangle $[0,R] \times [0, 2\pi]$ is mapped into a flat surface which is a disc in the xy-plane. In this case $\vec{r}_u \times \vec{r}_v = (\cos(v), \sin(v), 0) \times (-u\sin(v), u\cos(v), 0) = (0,0,u)$ and $|\vec{r}_u \times \vec{r}_v| = u = r$. We can explain the integration factor r in polar coordinates as a special case of a surface integral.

THE AREA OF THE SPHERE.

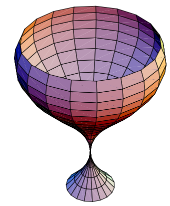
The map $\vec{r} : (u,v) \mapsto (L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v))$ maps the rectangle $R : [0, 2\pi] \times [0, \pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2 |\sin(v)|$ and $\int\int_R 1 \, dS = \int_0^{2\pi} \int_0^{\pi} L^2 \sin(v) \, dvdu = 4\pi L^2$.

SURFACE AREA OF GRAPHS. For surfaces $(u,v) \mapsto (u, v, f(u,v))$, we have $\vec{r}_u = (1, 0, f_u(u,v))$ and $\vec{r}_v = (0, 1, f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1 + f_u^2 + f_v^2}$. The area of the surface above a region R is $\int\int_R \sqrt{1 + f_u^2 + f_v^2} \, dA$.

EXAMPLE. The surface area of the paraboloid $z = f(x,y) = x^2 + y^2$ is (use polar coordinates) $\int_0^{2\pi} \sqrt{1 + 4r^2} r \, drd\theta = 2\pi(2/3)(1 + 4r^2)^{3/2}/8|_0^1 = \pi(5^{3/2} - 1)/6$.



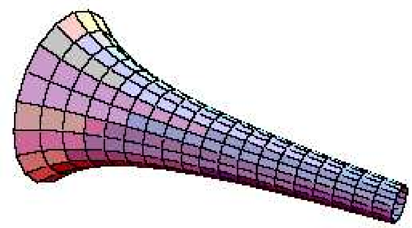
AREA OF SURFACES OF REVOLUTION. If we rotate the graph of a function $f(x)$ on an interval $[a,b]$ around the x-axes, we get a surface parameterized by $(u,v) \mapsto \vec{r}(u,v) = (v, f(v)\cos(u), f(v)\sin(u))$ on $R = [0,\pi] \times [a,b]$ and is called a **surface of revolution**. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$ which has the length $|\vec{r}_u \times \vec{r}_v| = |f(v)|\sqrt{1+f'(v)^2}\ dudv$.



EXAMPLE. If $f(x) = x$ on $[0,1]$, we get the surface area of a cone: $\int_0^{2\pi} \int_0^1 x\sqrt{1+1}\ dvdu = 2\pi\sqrt{2}/2 = \pi\sqrt{2}$.

P.S. In computer graphics, surfaces of revolutions are constructed from a few prescribed points $(x_i, f(x_i))$. The machine constructs a function (**spline**) and rotates

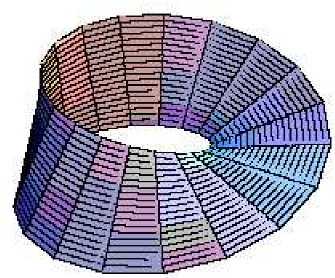
GABRIEL'S TRUMPET. Take $f(x) = 1/x$ on the interval $[1,\infty)$.
Volume: The volume is (use cylindrical coordinates in the x direction): $\int_1^\infty \pi f(x)^2\ dx = \pi \int_1^\infty 1/x^2\ dx = \pi$.
Area: The area is $\int_0^{2\pi} \int_1^\infty 1/x\sqrt{1+1/x^4}\ dx \geq 2\pi \int_1^\infty 1/x\ dx = 2\pi \log(x)|_0^\infty = \infty$.



The Gabriel trumpet is a surface of finite volume but with infinite surface area! You can fill the trumpet with a finite amount of paint, but this paint does not suffice to cover the surface of the trumpet!

Question. How long does a Gabriel trumpet have to be so that its surface is $500cm^2$ (area of sheet of paper)? Because $1 \leq \sqrt{1+1/x^4} \leq \sqrt{2}$, the area for a trumpet of length L is between $2\pi \int_1^L 1/x\ dx = 2\pi\log(L)$ and $\sqrt{2}2\pi\log(L)$. In our case, L is between $e^{500/(\sqrt{2}2\pi)} \sim 2 * 10^{24}cm$ and $e^{500/(2\pi)} \sim 4 * 10^{34}cm$. Note that the universe is about 10^{26} cm long (assuming that the universe expanded with speed of light since 15 Billion year). It could not accommodate a Gabriel trumpet with the surface area of a sheet of paper.

MÖBIUS STRIP. The surface $\vec{r}(u,v) = (2 + v\cos(u/2)\cos(u), (2 + v\cos(u/2))\sin(u), v\sin(u/2))$ parametrized by $R = [0,2\pi] \times [-1,1]$ is called a **Möbius strip**.
The calculation of $|\vec{r}_u \times \vec{r}_v| = 4 + 3v^2/4 + 4v\cos(u/2) + v^2\cos(u)/2$ is straightforward but a bit tedious. The integral over $[0,2\pi] \times [-1,1]$ is 17π.



QUESTION. If we build the Moebius strip from paper. What is the relation between the area of the surface and the weight of the surface?

REMARKS.
1) An OpenGL implementation of an Escher theme can be admired with
"xlock -inwindow -mode moebius"
on an X-terminal. 2) A patent was once assigned to the idea to use a Moebius strip as a **conveyor belt**. It would last twice as long as an ordinary one.