

Homework: Section 12.3: 10, 14, 22,

Section 12.4: 10, 12

THINGS TO KEEP IN MIND.

- Double integrals can often be evaluated through iterated integrals

"Integrals have layers".

- $\int \int_R 1 \, dx dy$ is the **area** of the region R .
- $\int \int_R f(x, y) \, dx dy$ is the volume of the solid having the graph of f as the "roof" and the R in the $x - y$ plane as the floor.



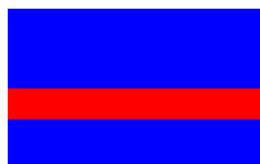
TYPES OF REGIONS.

$$\int \int_R f \, dA = \int_a^b \int_c^d f(x, y) \, dy dx$$
 rectangle.

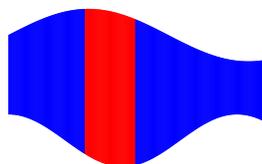
$$\int \int_R f \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy dx$$
 type I region.

$$\int \int_R f \, dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx dy$$
 type II region.

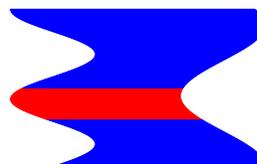
A general region, we try to cut it into pieces, where each piece is a Type I or Type II region.



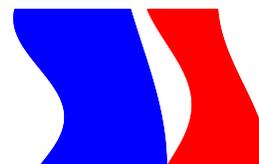
Rectangle



Type I



Type II



To cut

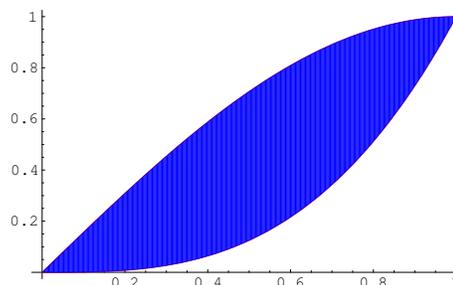
PROBLEM. Consider the type II integral

$$\int \int_R y \, dx dy = \int_0^1 \int_{2 \arcsin(y)/\pi}^{y^{1/3}} y \, dx dy .$$

Draw the region R , write the integral as a type I integral and evaluate the later.

Solution. The curves $x = \frac{2 \arcsin(y)}{\pi}$ and $x = y^{1/3}$ bounding the region are identical to the curves $y = \sin(\pi x/2)$ and $y = x^3$.

The region is bounded by these two curves:

The corresponding type I integral is $\int_0^1 \int_{x^3}^{\sin(\pi x/2)} y \, dy dx = \int_0^1 \sin(\pi x/2)^2/2 - x^6/2 \, dx$.

POLAR COORDINATES. For many regions, it is better to use polar coordinates for integration:

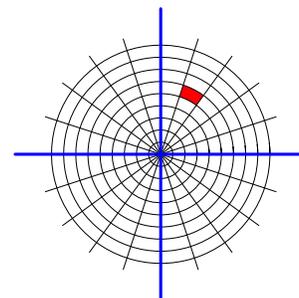
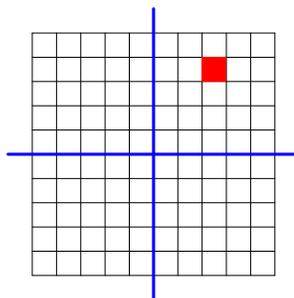
$$\int \int_R f(x, y) \, dx dy = \int \int_R f(r, \theta) r \, dr d\theta$$

EXAMPLE. The area of the disc $\{x^2 + y^2 \leq 1\}$ can be computed by treating the region as a type I region and doing the integral with $x = \sin(u)$, $dx = \cos(u)du$: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dydx = \int_{-1}^1 2\sqrt{1-x^2} \, dx = \int_{-\pi/2}^{\pi/2} 2 \cos^2(u) \, du = \pi$. It is easier to do that integral in polar coordinates:

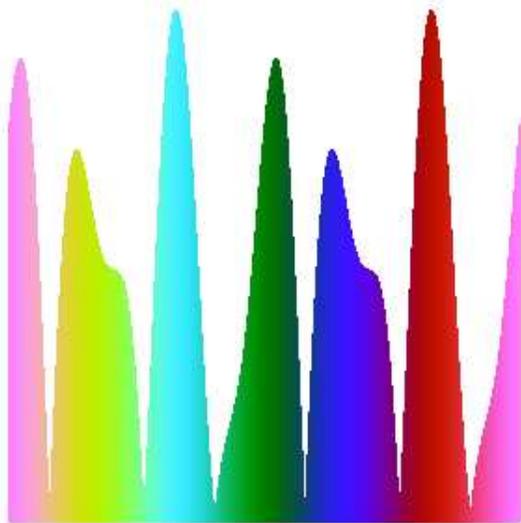
$$\int_0^{2\pi} \int_0^1 r \, drd\theta = 2\pi r^2/2|_0^1 = \pi .$$

WHERE DOES THE FACTOR "r" COME FROM?

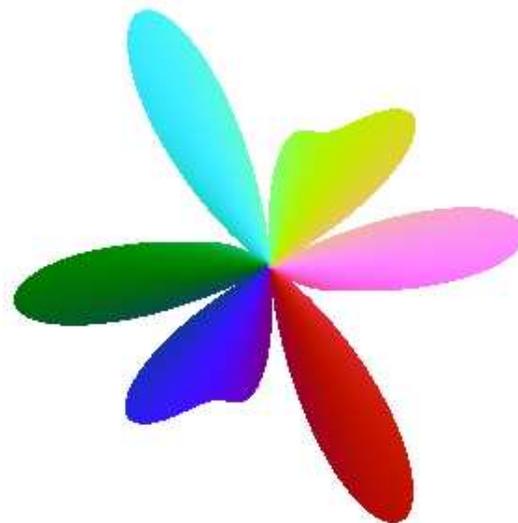
A small rectangle with dimensions $d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta) \mapsto (r \cos(\theta), r \sin(\theta))$ to a sector segment in the (x, y) plane. It has approximately the area $r d\theta dr$. It is small for small r .



ROSES. We can now integrate over type I or type II regions in the (θ, r) plane. Examples are **roses**: $\{(\theta, r) \mid 0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ .



The region R in the $\theta - r$ coordinates is a type I region



The region R in the $x - y$ coordinates is neither a type I nor a type II region.

EXAMPLE. Find the area of the region $\{(\theta, r(\theta)) \mid r(\theta) \leq |\cos(3\theta)|\}$.

$$\iint_R y \, dx dy = \int_0^{2\pi} \int_0^{|\cos(3\theta)|} r \, dr d\theta = \int_0^{2\pi} \frac{\cos^2(3\theta)}{2} \, d\theta = \pi/2$$

EXAMPLE. Integrate $f(x, y) = y\sqrt{x^2 + y^2}$ over the semi annulus $R = \{(x, y) \mid 1 < x^2 + y^2 < 4, y > 0\}$.

Solution.

$$\int_1^2 \int_0^\pi r \sin(\theta) r r \, d\theta dr = \int_1^2 r^3 \int_0^\pi \sin(\theta) \, d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^\pi \sin(\theta) \, d\theta = 15/2$$

For integration problems, where the region is part of an annulus, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r \cos(\theta)$, $y = r \sin(\theta)$.