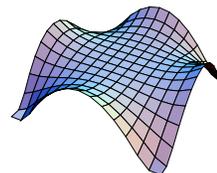


PARTIAL DERIVATIVE. If $f(x, y, z)$ is a function of three variables, then $\frac{\partial}{\partial x}f(x, y, z)$ is defined as the derivative of the function $g(x) = f(x, y, z)$, where y and z are fixed. The other derivatives with respect to y and z are defined similarly.

REMARK. The partial derivatives measure the rate of change of the function in the x, y , or z directions.

NOTATION. One also writes $f_x(x, y, z) = \frac{\partial}{\partial x}f(x, y, z)$ etc. For iterated derivatives the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f$.

EXAMPLE. $f(x, y) = x^4 - 6x^2y^2 + y^4$. We have $f_x(x, y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x, y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$. We see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is called **harmonic**. The equation itself is called a **partial differential equation** (see separate handout).



CLAIROT THEOREM. If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$. Proof. Compare the two sides:

$$dx f_x(x, y) \sim f(x + dx, y) - f(x, y)$$

$$dy dx f_{xy}(x, y) \sim f(x + dx, y + dy) - f(x + dx, y) - (f(x + dx, y) - f(x, y))$$

$$dy f_y(x, y) \sim f(x, y + dy) - f(x, y)$$

$$dx dy f_{yx}(x, y) \sim f(x + dx, y + dy) - f(x, y + dy) - (f(x, y + dy) - f(x, y))$$

CONTINUITY IS NECESSARY. Example: $f(x, y) = (x^3y - xy^3)/(x^2 + y^2)$ contradicts Clairot:

$$f_x(x, y) = (3x^2y - y^3)/(x^2 + y^2) - 2x(x^3y - xy^3)/(x^2 + y^2)^2, f_x(0, y) = -y, f_{xy}(0, 0) = -1,$$

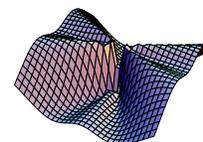
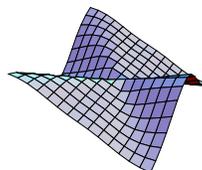
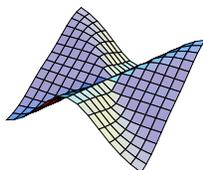
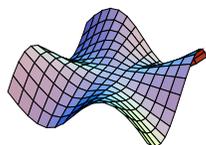
$$f_y(x, y) = (x^3 - 3xy^2)/(x^2 + y^2) - 2y(x^3y - xy^3)/(x^2 + y^2)^2, f_y(x, 0) = x^2, f_{yx}(0, 0) = 1.$$

$$f(x, y)$$

$$f_x(x, y)$$

$$f_y(x, y)$$

$$f_{xy}(x, y)$$

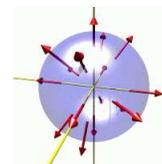


GRADIENT. If $f(x, y, z)$ is a function of three variables, then

$$\nabla f(x, y, z) = \left(\frac{\partial}{\partial x}f(x, y, z), \frac{\partial}{\partial y}f(x, y, z), \frac{\partial}{\partial z}f(x, y, z) \right)$$

is called the **gradient** of f . The symbol ∇ is called **Nabla**.

NORMAL. As we will see later, the gradient $\nabla f(x, y, z)$ is orthogonal to the level curve $f(x, y, z) = c$ and the gradient $\nabla f(x, y, z)$ is normal to the level surface $f(x, y, z)$. For example, the gradient of $f(x, y, z) = x^2 + y^2 - z^2$ at a point (x, y, z) is $(2x, 2y, -2z)$.



- Geometry. For example, the gradient $\nabla f(x, y, z)$ is a vector normal to a surface at the point (x, y, z) . Tangent spaces.
- Approximations, linearizations.
- Partial differential equations. Laws which describe physics.
- Optimization problems, as we will see later.
- Solution to some integration problems using generalizations of fundamental theorem of calculus.
- Generally helpful to understand and analyze functions of several variables.

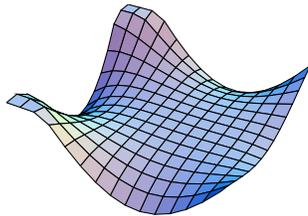
PARTIAL DIFFERENTIAL EQUATIONS. An equation which involves partial derivatives of an unknown function is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

1) $f_{xx}(x, y) = f_{yy}(x, y)$ is an example of a partial differential equation 1) $f_x(x, y) = f_{xx}(x, y)$ would be an ordinary differential equation (the variable y can be considered as a parameter).

LAPLACE.

$$\boxed{f_{xx} + f_{yy} = 0}.$$

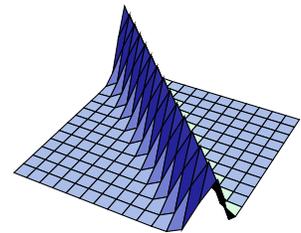
$$f(x, t) = x^2 - y^2.$$



ADVECTION.

$$\boxed{f_t = f_x}.$$

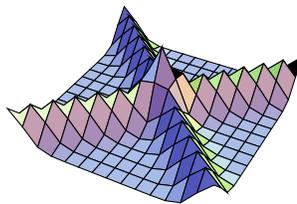
$$f(x, t) = g(x - t).$$



WAVES.

$$\boxed{f_{tt} = f_{xx}}.$$

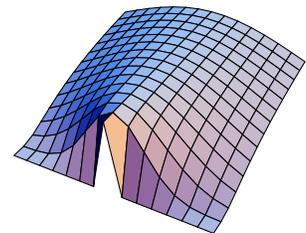
$$f(t, x) = \sin(x - t) + \sin(x + t)$$



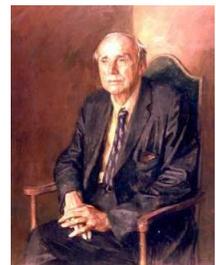
HEAT

$$\boxed{f_t = f_{xx}}$$

$$f(t, x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$$



"A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." - Paul A. M. Dirac.



Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an **antiparticle**: the existence of antiparticles was later confirmed.