HOMEWORK: Section 13.3: problems 6,16,20,22,26

REVIEW.

If F is a vector field and $\gamma : t \mapsto \vec{r}(t)$ is a curve, then $\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$ is called the **line integral** of F along the curve γ.

A vector field $F = \nabla f$ is called a **gradient field** or **conservative vector field**.

The **chain rule** tells that $d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.

FUNDAMENTAL THEOREM OF LINE INTEGRALS. If $F = \nabla f$, then

$$\int_a^b F(r(t)) \cdot r'(t)\, dt = f(r(b)) - f(r(a))$$

The line integral gives the potential difference between the points $r(b)$ and $r(a)$.

EXAMPLE. Let $f(x, y, z)$ be the temperature distribution in a room and let $r(t)$ the path of a fly in the room, then $f(r(t))$ is the temperature, the fly experiences at the point $r(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(r(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end and initial point.

SPECIAL CASES.
$r(t)$ parallel to level curve means $d/dt f(r(t)) = 0$ and $r'(t)$ orthogonal to $\nabla f(r(t))$
$r(t)$ orthogonal to level curve means $|d/dt f(r(t))| = |\nabla f||r'(t)|$ and $r'(t)$ parallel to $\nabla f(r(t))$.

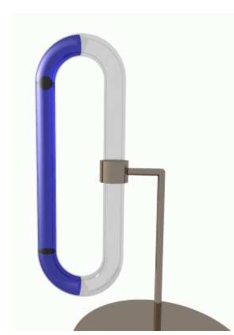
PROOF OF THE FUNDAMENTAL THEOREM. Use the chain rule in the second equality and the fundamental theorem of calculus in the third equality:

$$\int_a^b F(r(t)) \cdot r'(t)\, dt = \int_a^b \nabla f(r(t)) \cdot r'(t)\, dt = \int_a^b \frac{d}{dt} f(r(t))\, dt = f(r(b)) - f(r(a)) .$$

CLOSED CURVES. We see from the last example that the line integral along a closed curve is zero if the vector field is a gradient field. The work done along a closed path is zero. This is a form of **energy conservation**.

If F is a gradient field, then the line integral along a closed curve is zero. The line integral is in that case also independent of paths.

PERPETUUM MOTION MACHINES. A machine which implements a force field which is not a gradient field is called a **perpetuum mobile**. Mathematically, it realizes a force field for which there exist some closed loops along which the energy gain is nonnegative. (By possibly changing the direction, the energy change can be made positive). The first law of thermodynamics forbids the existence of such a machine.
It is informative to stare at some of the ideas people have come up with and to see why they don't work. The drawings of Escher appear also to produce situations, where a force field can be used to gain energy. Escher uses genius graphical tricks however.



WHEN IS A VECTOR FIELD A GRADIENT FIELD (2D)?

$F(x, y) = \nabla f(x, y)$ implies $F_y(x, y) = F_x(x, y)$. If this does not hold at some point, $F = (P, Q)$ is no gradient field. We will see soon why the condition $\text{curl}(F) = Q_x - P_y = 0$ assures that F is conservative.

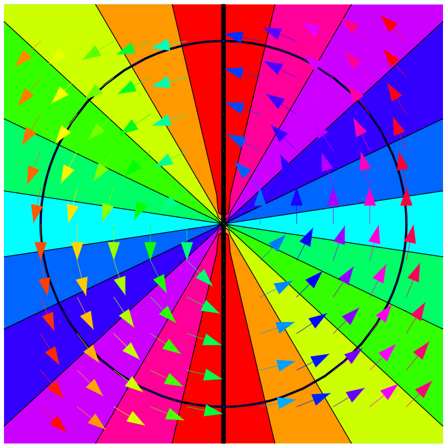
PROBLEM 1. Let $F(x,y) = (2xy^2 + 3x^2, 2y)$. Find a potential f of F.

SOLUTION. The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$.

Find G in $F = (P,Q) = \nabla f$ from $f(x,y) = \int_0^x P(x,y)\,dx + h(y)$ and finding h from $f_y(x,y) = g(x,y)$.

PROBLEM 2. Find values for the constants a,b which make the vector field $F = (P,Q) = (ax^3y + by^2, x^4 + yx)$ conservative.
SOLUTION. Compute $Q_x - P_y = 4x^3 + y - ax^3$ which must vanish so that $a = 4$ and $b = 1/2$. The potential is $f = \nabla(x^4 + y^2x/2)$.

A CONSERVATIVE VECTORFIELD? Let $F(x,y) = (P,Q) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = P, f_y = (1/x)/(1+y^2/x^2) = Q$.



The line integral $\int_\gamma F \quad ds$, where γ is the unit circle is $\int_0^{2\pi} (\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2}) \cdot (-\sin(t), \cos(t)\ dt = \int_0^{2\pi} 1\ dt = 2\pi$.

There is no contradiction to the fundamental theorem of line integrals because the potential f as well as the vectorfield F are not smooth everywhere.

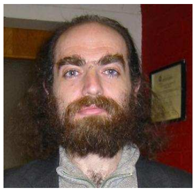
SIMPLY CONNECTED. A region R is called **simply connected**, if every curve in that region can be contracted to a point in a continuous way and every two points can be connected by a path.

A disc is an example of a simply connected region, an annulus is an example of a not simply connected region.

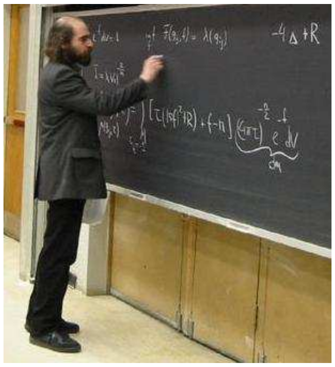
PRECICE REFORMULATION OF THE CLOSED CURVE STATEMENT:

If R is a simply connected region and F is a gradient field $F = \nabla f$ with smooth f, then $\int_\gamma F\ dr = 0$ for every closed curve in R.

MATHEMATICS IN THE NEWS. This week, the New York times reported from a new attempt by Grigori Perelman to an old and famous unsolved problem in Mathematics, the Poincaré conjecture. (The photos of an event at MIT http://www-math.mit.edu/conferences/simons/photoalbum.html were also mentioned in slashdot.) The Poincaré conjecture asks



Every bounded three dimensional and simply connected space can be deformed into the three dimensional sphere.



In more technical terms, mathematicians say: every simply connected compact 3-manifold without boundary is homeomorphic to a 3-sphere. The Clay Math institute states the conjecture for Laymen: "If we stretch a rubber band around the surface of an apple, then we can shrink it down to a point by moving it slowly, without tearing it and without allowing it to leave the surface. On the other hand, if we imagine that the same rubber band has somehow been stretched in the appropriate direction around a doughnut, then there is no way of shrinking it to a point without breaking either the rubber band or the doughnut. We say the surface of the apple is "simply connected," but that the surface of the doughnut is not. Poincar; almost a hundred years ago, knew that a two dimensional sphere is essentially characterized by this property of simple connectivity, and asked the corresponding question for the three dimensional sphere. This question turned out to be extraordinarily difficult, and mathematicians have been struggling with it ever since."