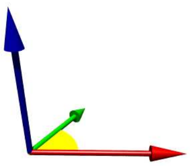
2/7/2003, CROSS PRODUCT **Math 21a, O. Knill**

Homework for Monday: Section 9.4, Numbers 16, 18, 26, 32, 34

CROSS PRODUCT. The **cross product** of two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is defined as the vector $\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$.



To compute it: multiply diagonally at the crosses.

v_1	v_2		v_3		v_1		v_2	
		X		X		X		
w_1	w_2		w_3		w_1		w_2	

DIRECTION OF $\vec{v} \times \vec{w}$: $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and orthogonal to $\vec{w}$.

Proof. Check that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$.

LENGTH: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$

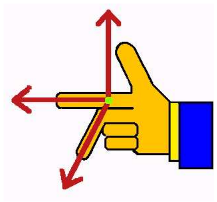
Proof. The identity $|v \times w|^2 = |v|^2|w|^2 - (v \cdot w)^2$ can be proven by direct computation. Now, $|v \cdot w| = |v||w| \cos(\alpha)$.

AREA. The length $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by v and w.

Proof. Check it first for $v = (1, 0, 0)$ and $w = (\cos(\alpha), \sin(\alpha), 0)$, where $v \times w = (0, 0, \sin(\alpha))$ has length $|\sin(\alpha)|$ which is indeed the area of the parallelogram spanned by v and w. A more general case can be obtained by scaling v and w: both the area as well as the cross product behave linearly in v and w.

ZERO CROSS PRODUCT. We see that $\vec{v} \times \vec{w}$ is zero if $\vec{v}$ and $\vec{w}$ are parallel.

ORIENTATION. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. The right hand rule is: put the first vector $\vec{v}$ on the thumb, the second vector $\vec{w}$ on the pointing finger and the third vector $\vec{v} \times \vec{w}$ on the third middle finger.

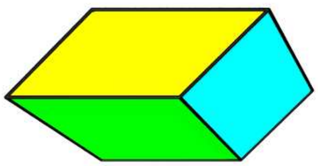


EXAMPLE. $\vec{i} \times \vec{j} = \vec{k}$ forms a right handed coordinate system.

DOT PRODUCT (is a scalar)		CROSS PRODUCT (is a vector)	
$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$	commutative	$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$	anti-commutative
$\|\vec{v} \cdot \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \cos(\alpha)$	angle	$\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \sin(\alpha)$	angle
$(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$	linearity	$(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})$	linearity
$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$	distributivity	$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$	distributivity
{1, 2, 3}.{3, 4, 5}	in Mathematica	Cross[{1, 2, 3}, {3, 4, 5}]	in Mathematica
$\frac{d}{dt}(\vec{v} \cdot \vec{w}) = \dot{v} \cdot \vec{w} + \vec{v} \cdot \dot{w}$	product rule	$\frac{d}{dt}(\vec{v} \times \vec{w}) = \dot{\vec{v}} \times \vec{w} + \vec{v} \times \dot{\vec{w}}$	product rule

TRIPLE SCALAR PRODUCT. The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot \vec{v} \times \vec{w}$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$.

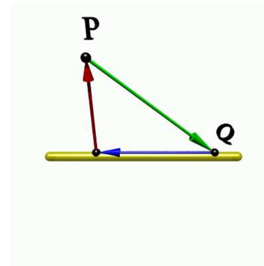
PARALLELEPIPED. $[\vec{u}, \vec{v}, \vec{w}]$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ because $h = \vec{u} \cdot \vec{n}/|\vec{n}|$ is the height of the parallelepiped if $\vec{n}$ is a normal vector to the ground parallelogram which has area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $Ah = |u \cdot (\vec{v} \times \vec{w})|$.



DISTANCE POINT-LINE (3D).
If P is a point in space and L is the line which contains the vector $\vec{u}$, then

$$d(P,L) = |(\vec{PQ} \times \vec{u}|/|\vec{u}|$$

is the distance between P and the line L.



PLANE THROUGH 3 POINTS P, Q, R:

The vector $= \vec{PQ} \times \vec{PR}$ is orthogonal to the plane. We will see this again next week.

The rest is informal and serves only as a motivation for this course:

ANGULAR MOMENTUM. If a mass point of mass m moves along a curve $\vec{r}(t)$, then the vector $\vec{L}(t) = m\vec{r}(t) \times \vec{r}'(t)$ is called the **angular momentum**.

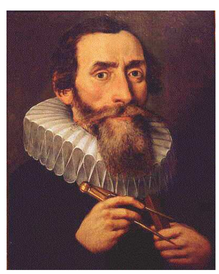
ANGULAR MOMENTUM CONSERVATION.

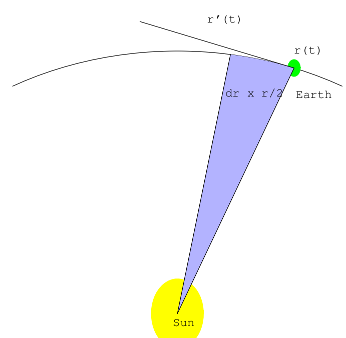
$$\frac{d}{dt}\vec{L}(t) = m\vec{r}'(t) \times \vec{r}'(t) + m\vec{r}(t) \times \vec{r}''(t) = \vec{r}(t) \times \vec{F}(t)$$

In a central field, where $\vec{F}(t)$ is parallel to $\vec{r}(t)$, this vanishes.

TORQUE. In physics, the quantity $\vec{r}(t) \times \vec{F}(t)$ is also called the **torque**. The time derivative of the **momentum** $m\vec{r}'$ is the **force**, the time derivative of the **angular momentum** $\vec{L}$ is the **torque**.

KEPLER'S AREA LAW. (Proof by Newton) The fact that $\vec{L}(t)$ is constant means first of all that $\vec{r}(t)$ stays in a plane spanned by $\vec{r}(0)$ and $\vec{r}'(0)$. The experimental fact that the vector $\vec{r}(t)$ sweeps over **equal areas in equal times** expresses the angular momentum conservation: $|\vec{r}(t) \times \vec{r}'(t)dt/2| = |\vec{L}dt/m/2|$ is the area of a small triangle. The vector $\vec{r}(t)$ sweeps over an area $\int_0^T |\vec{L}|dt/(2m) = |\vec{L}|T/(2m)$ in time $[0,T]$.





PLACES IN PHYSICS WHERE THE CROSS PRODUCT OCCURS: (informal)

The **top**, the motion of a rigid body is describe by the angular momentum L and the angular velocity vector Ω in the body. Then $\dot{L} = L \times \Omega + M$, where M is an external torque.

Electromagnetism: a particle moving along $\vec{r}(t)$ in a **magnetic field** $\vec{B}$ for example experiences the force $\vec{F}(t) = q\vec{r}'(t) \times \vec{B}$, where q is the charge of the particle.

Hurricanes are powerful storms with wind velocities of 74 miles per hour or more. On the northern hemisphere, hurricanes turn counterclockwise, on the southern hemisphere clockwise. This is a feature of all low pressure systems and can be explained by the Coriolis force. In a rotating coordinate system a particle of mass m moving along $\vec{r}(t)$ experience the following forces: $m\vec{\omega}' \times \vec{r}$ (inertia of rotation), $2m\vec{\omega} \times \vec{r}'$ (Coriolis force) and $m\omega \times (\vec{\omega} \times \vec{r}))$ (Centrifugal force), a fundamental physical force which is also responsible for the circulation in Jupiter's Red Spot.