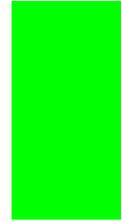
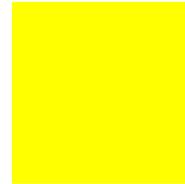
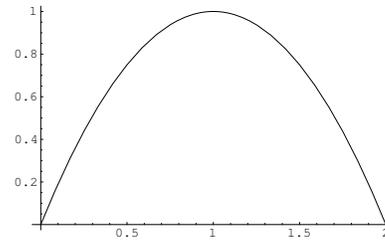


HOMEWORK: Section 11.8: 6, 12, 18, 19, 36

AREA OF SQUARE. We want to extremize the area of a rectangle for which the length of the boundary is fixed 4. If the sides are x and y , then we want to extremize $f(x, y) = xy$ under the constraint $g(x, y) = 2x + 2y = 4$. The Lagrange equations $y = 2\lambda, x = 2\lambda$ show that $x = y$ and so $x = y = 1$.



The last problem could also be solved by substituting $y = 2 - x$ into the area formula $A = xy = x(2 - x)$ leading to a one-dimensional extremal problem: maximize $f(x) = x(2 - x)$ on the interval $[0, 2]$. To do so, we have to find the extrema inside the interval and then consider also the boundary points $x = 0, x = 2$. Again, we get $x = 1$.



VOLUME OF CUBE. Extremize the volume $f(x, y, z) = xyz$ of a box with fixed surface area $xy + yz + xz = 3$.

To solve $yz = \lambda(y + z), xz = \lambda(x + z), xy = \lambda(x + y), xy + yz + xz = 1$, take quotients: $z/x = (y + z)/(y + x), z/y = (z + x)/(y + x)$ which gives $z(y + x) = x(y + z), z(y + x) = y(z + x)$ so that either $xz = yz$ or $z = 0$. Similarly, we get $y = z$ or $y = 0$. The solution is $x = y = z = 1$.

AN OTHER SOLUTION. For a solution without Lagrange multipliers, we would plug in $z = (1 - xy)/(y + x)$ and try to find the maximum of $f(x, y) = xy(1 - xy)/(y + x)$ on the domain $D = \{x > 0, y > 0, xy \leq 1\}$.

We first would have to find critical points inside the region D :

$$f_x(x, y) = -y(1 - 2xy)/(x + y) - xy(1 - xy)/(x + y)^2 = 0$$

$$f_y(x, y) = -x(1 - 2xy)/(x + y) - xy(1 - xy)/(x + y)^2 = 0$$

The difference of these two equations gives $(x - y)(1 - 2xy) = 0$ so that either $x = y$ or $xy = 1/2$. The second case can not give us $f_x = f_y = 0$. The first condition $x = y$ gives $x = y = 1$ which is not inside the region. However, on the boundary $g(x, y) = xy = 1$, the Lagrange equations $\nabla f = \lambda \nabla g$ have a solution with $(x, y) = (1, 1)$.

This example illustrates the power of Lagrange multipliers. The solution using substitution is much more complicated.

GENERAL PROBLEM. Given a region G whose boundary is given by $g(x, y) = c$. The task to maximize or minimize $f(x, y)$ on G has the following steps:

I) Find extrema inside the region: compute critical points $\nabla f = (0, 0)$ and classify them using the second derivative test.

II) Find extrema on the boundary using Lagrange: $\nabla f = \lambda \nabla g, g = c$.

III) Compare the values of the functions obtained in I) and II) to find the maximum or minimum.

EXAMPLE. Extremize $f(x, y) = 3x^2 - 4x - y^2$ on $x^2 + y^2 = 1$.

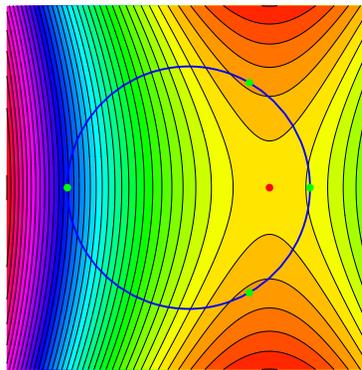
I) Find the extrema inside the disc. There is only one critical point $(2/3, 0)$. The discriminant D is -6 so that $(2/3, 0)$ is a saddle point.

II) On the boundary, we have to solve $6x - 4 = 2\lambda x, -2y = 2\lambda y$. There are four solutions: $(1/2, -\sqrt{3}/2), (1/2, +\sqrt{3}/2), (1, 0), (-1, 0)$.

III) We make a list of all the candidates:

(x, y)	$f(x, y)$
$(2/3, 0)$	$-4/3$
$(1/2, -\sqrt{3}/2)$	-2
$(1/2, \sqrt{3}/2)$	-2
$(1, 0)$	-1
$(-1, 0)$	7

and see that $(-1, 0)$ is the maximum and $(1/2, -\sqrt{3}/2)$ are the minima.



IN MATHEMATICA.

Here is how a machine solves the above problem. After defining the functions f and g , the machine solves first the equations leading to critical points, and then the Lagrange equations (we put $L = \lambda$).

$$f[x_, y_] := 3x^2 - 4x - y^2$$

$$g[x_, y_] := x^2 + y^2 - 1$$

$$\text{Solve}\{D[f[x, y], x] == 0, D[f[x, y], y] == 0\}, \{x, y\}$$

$$\text{Solve}\{D[f[x, y], x] == L * D[g[x, y], x], D[f[x, y], y] == L * D[g[x, y], y], g[x, y] == 0\}, \{x, y, L\}$$

AN OTHER LAGRANGE PROBLEM. (A bit tricky).

PROBLEM. Let p and q be positive constants such that $\frac{1}{p} + \frac{1}{q} = 1$. Use the method of Lagrange Multipliers to prove that for any $x > 0, y > 0$, the following inequality is true:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

SOLUTION.

We have to find the maximum of $f(x, y) = xy$ ($x > 0, y > 0$) under the constraint $\frac{x^p}{p} + \frac{y^q}{q} = c$.

The Lagrange equations $x = \lambda x^{p-1}, y = \lambda y^{q-1}$ gives $y/x = x^{p-1}/y^{q-1}$ so that $y^q = x^p$.

Plugging this into $x^p/q + y^q = c$ gives $x^p(1/p + 1/q) = c$ or $x = c^{1/p}$ and so $y = c^{1/q}$. The maximal value of $f(x, y) = xy$ is $c^{1/p}c^{1/q} = c^1 = c$. Therefore, everywhere

$$xy = f(x, y) \leq c = x^p/p + y^q/q.$$