4/11/2003, CHANGE OF COORDINATES — Math 21a, O. Knill

HOMEWORK. Section 12.9: 4, 10, 22. Section 12.8: 30, 16

CHANGE OF COORDINATES 1D. (Substitution) $x = T(u), dx = T'(u)du$ for 1D definite integrals:

$$\int_{T(a)}^{T(b)} f(x)\, dx = \int_a^b f(T(u)) \frac{\partial x}{\partial u}\, du$$

Example:
$\int_{-1}^{1} \sqrt{1-x^2} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u) du$, $x = T(u) = \sin(u), dx = T'(u)du = \cos(u)du, f(T(u)) = \sqrt{1-\sin^2(u)}$.

JACOBEAN 2D. A map $T(u,v) = (x(u,v), y(u,v))$ defines a map on the plane. It can serve as a coordinate transformation. ¡br¿
The Jacobean of a transformation $T(u,v) = (x(u,v), y(u,v))$ is defined as
$\frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$.

PROOF.
If $\vec{r}(u,v) = (x(u,v), y(u,v), 0)$ parameterizes a surface, then $|\vec{r}_u \times \vec{r}_v| = \frac{\partial(x,y)}{\partial(u,v)}$. By extending the map to a map from the plane to space, we can identify the Jacobean as a surface area element, which we are familiar with.

CHANGE OF COORDINATES 2D.

$$\int\int_{T(R)} f(x,y)\, dxdy = \int\int_R f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, dudv.$$

JACOBEAN 3D. The Jacobean of a transformation $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ is denoted by $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ and defined as the triple product of the three vectors $\nabla x(u,v,w)$, $\nabla y(u,v,w)$ and $\nabla z(u,v,w)$.
(This is the volume of the parallelepiped spanned by these three vectors.)

SUBSTITUTION IN THREE DIMENSIONS If $(x,y,z) = T(u,v,z)$ is an arbitrary change of variables, the formula

$$\int\int\int_{T(R)} f(x,y,z)\, dxdydz = \int\int\int_R f(u,v,w)\, \frac{\partial(x,y,z)}{\partial(u,v,w)}\, dudvdw$$

holds.

JACOBEAN AS DETERMINANT. (More in Math21b). One can actually see that the Jacobean is a determinant of a matrix $T'(\vec{u})$ attached to each point. It generalizes to any dimension. Here is an overview in dimensions $1, 2$ and 3:

1D	$T(u) = x(u)$	$T'(u) = \left[\, x'(u) \,\right]$.
2D	$T(u,v) = (x(u,v), y(u,v))$	$T'(u,v) = \begin{bmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \end{bmatrix}$.
3D	$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$	$T'(u,v,w) = \begin{bmatrix} x_u(u,v,w) & x_v(u,v,w) & x_w(u,v,w) \\ y_u(u,v,w) & y_v(u,v,w) & y_w(u,v,w) \\ z_u(u,v,w) & z_v(u,v,w) & z_w(u,v,w) \end{bmatrix}$

The Jacobean is the determinant in each case.

EXAMPLE POLAR COORDINATES. Here $(u,v) = (r,\theta)$.
$T(u,v) = (u\cos(v), u\sin(v))$.
$T'(u,v) = \begin{bmatrix} \cos(v) & -u\sin(v) \\ \sin(v) & u\cos(v) \end{bmatrix}$. $\frac{\partial x,y}{\partial u,v} = u$. This is the factor r we have seen already.

EXAMPLE CYLINDRICAL COORDINATES. Here $(u,v,w) = (r,\theta,z)$.
$T(u,v,w) = (u\cos(v), u\sin(v), w)$. $T'(u,v,w) = \begin{bmatrix} \cos(v) & -u\sin(v) & 0 \\ \sin(v) & u\cos(v) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $\frac{\partial x,y}{\partial u,v} = u$.

EXAMPLE SPHERICAL COORDINATES. Here $(u, v, w) = (r, \theta, \phi)$.
$T(u, v, w) = (u\cos(v)\sin(w), u\sin(v)\sin(w), u\cos(w))$.

$$T'(u, v, w) = \begin{bmatrix} \cos(v)\sin(w) & -u\sin(v)\sin(w) & u\cos(v)\cos(w) \\ \sin(v)\sin(w) & u\cos(v)\sin(w) & u\sin(v)\cos(w) \\ \cos(w) & 0 & u\sin(w) \end{bmatrix}.$$

$\frac{\partial x,y,z}{\partial u,v,w} = u^2\sin(w)$. This is the factor $\rho^2\sin(\phi)$ in the old notation.

EXAMPLE SCALING. If $T(u, v, w) = (au, bv, cw)$, then this corresponds to a distortion in the x, y and z direction.

$$T'(u, v, w) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$ The Jacobian is abc.

APPLICATION.
Task: Calculate the volume of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

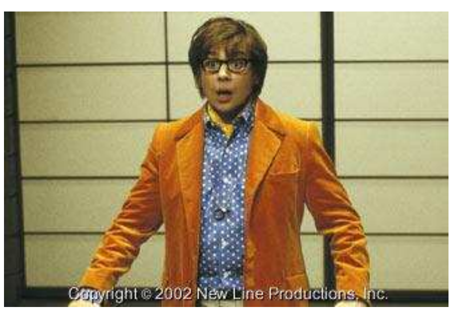
Solution: The ellipsoid E is the image of the transformation $(x, y, z) = T(u, v, w) = (au, bv, cw)$ because if $u^2 + v^2 + c^2 = 1$, then $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. The Jacobean is abc. So, the volume of the ellipsoid is abc times the volume of the unit sphere which is $\boxed{4\pi abc/3}$.

EXAMPLE ELLIPTICAL COORDINATES. Here $(u, v, w) = (r, \theta, \phi)$.
The spherical and scaling coordinate system lead to a new coordinate system which is called **elliptical coordinate system**. $T(u, v, w) = (au\cos(v)\sin(w), bu\sin(v)\sin(w), cu\cos(w))$.

$$T'(u, v, w) = \begin{bmatrix} a\cos(v)\sin(w) & -au\sin(v)\sin(w) & au\cos(v)\cos(w) \\ b\sin(v)\sin(w) & bu\cos(v)\sin(w) & bu\sin(v)\cos(w) \\ c\cos(w) & 0 & cu\sin(w) \end{bmatrix}.$$

$\frac{\partial x,y,z}{\partial u,v,w} = abcu^2\sin(w)$. This is the factor $abc\rho^2\sin(\phi)$ in the old notation of spherical coordinates.

EXAMPLE: IMAGE FILTERS.
In computer graphics, transformations form a subclass of Filters. The example shows Austin Powers mapped into inverse polar coordinates.



EXAMPLE TORAL COORDINATES. The coordinate change $T(u, v, w) = ((b + w\cos(v))\cos(u), (b + w\cos(v))\sin(u), w\sin(v))$ allows to describe the interior of a torus as the image of the rectangular solid $R : [0, 2\pi] \times [0, 2\pi] \times [0, a]$ (see seperate handout). The Jacobean is $w(b + w\cos(v))$. Integrating this over R gives $V = 2\pi^2a^2b$ as the volume of the torus.

AREA PRESERVING. A coordinate change is called **area preserving** if the Jacobean is 1.

EXAMPLES:
$T(u, v) = (u\cos(\theta) - v\sin(\theta), u\sin(\theta) + v\cos(\theta))$ **rotation**.
$T(u, v) = (u + f(w), v + g(w), w)$ **shear**. That this does not change the volume is a special case of the **Cavalieri Principle**.