

**Homework:** Section 11.5: 6, 18, 24, 28, 30

REMEMBER? If  $f$  and  $g$  are functions of one variable  $t$ , then  $d/dt f(g(t)) = f'(g(t))g'(t)$ . For example,  $d/dt \sin(\log(t)) = \cos(\log(t))/t$ .

THE CHAIN RULE. If  $\vec{r}(t)$  is curve in space and  $f$  is a function of three variables, we get a function of one variables  $t \mapsto f(\vec{r}(t))$ . The chain rule looks like the 1D chain rule, but where derivative  $f'$  is replaced with the gradient  $\nabla f = (f_x, f_y, f_z)$ . The derivative is  $d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

WRITING IT OUT. Writing the dot product out gives

$$\frac{d}{dt} f(\vec{r}(t)) = f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t).$$

In the case of two variables

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) \frac{dx(t)}{dt} + f_y(x(t), y(t)) \frac{dy(t)}{dt}$$

(see case 1 in the book).

EXAMPLE. Let  $z = \sin(x + 2y)$ , where  $x$  and  $y$  are functions of  $t$ :  $x = e^t$ ,  $y = \cos(t)$ . What is  $\frac{dz}{dt}$ ?

Here,  $z = f(x, y) = \sin(x + 2y)$ ,  $z_x = \cos(x + 2y)$ , and  $z_y = 2 \cos(x + 2y)$  and  $\frac{dx}{dt} = e^t$ ,  $\frac{dy}{dt} = -\sin(t)$  and  $\frac{dz}{dt} = \cos(x + 2y)e^x - 2 \cos(x + 2y) \sin(t)$ .

EXAMPLE. If  $f$  is the temperature distribution in a room and  $\vec{r}(t)$  is the path of a spider then  $f(\vec{r}(t))$  is the temperature, the spider experiences at time  $t$ . The rate of change depends on the velocity  $\vec{r}'(t)$  of the spider as well as the temperature gradient  $\nabla f$  and the angle between gradient and velocity. For example, if the spider moves perpendicular to the gradient, its velocity is tangent to a level curve and the rate of change is zero.



EXAMPLE. Assume the spider moves along a circle  $\vec{r}(t) = (\cos(t), \sin(t))$  on a table with temperature distribution  $T(x, y) = x^2 - y^3$ . Find the rate of change of the temperature, the spider experiences in time.

SOLUTION.  $\nabla T(x, y) = (2x, -3y^2)$ ,  $\vec{r}'(t) = (-\sin(t), \cos(t))$   $d/dt T(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = (2 \cos(t), -3 \sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2 \cos(t) \sin(t) - 3 \sin^2(t) \cos(t)$ .

APPLICATION: IMPLICIT DIFFERENTIATION.

1) **2D** From  $f(x, y) = 0$  one can express  $y$  as a function of  $x$ . From  $d/df(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$  we obtain  $y' = -f_x/f_y$ .

2) **3D** If  $z = g(x, y)$  is given by  $f(x, y, z) = 0$  then  $f_x + f_z g_x = 0$  and  $f_y + f_z g_y = 0$  so that  $g_x = -f_x/f_z$  and  $g_y = -f_y/f_z$ .

EXAMPLE.  $f(x, y) = x^4 + x \sin(xy) = 0$  defines  $y = g(x)$ . If  $f(x, g(x)) = 0$ , then  $g_x(x) = -f_x/f_y = -(4x^3 + \sin(xy) + xy \cos(xy))/(x^2 \cos(xy))$ .

APPLICATION: If  $f(x, y, z) = 0$ , then  $x = x(y, z)$ ,  $y = y(x, z)$  and  $z = z(x, y)$ . From  $y_x = -f_x/f_y$ ,  $x_z = -f_z/f_x$ ,  $z_y = -f_y/f_z$  we get the relation  $y_x x_z z_y = -1$ . This formula appears in thermodynamics.

**DIETERICI EQUATION.** In thermodynamics the temperature  $T$ , the pressure  $p$  and the volume  $V$  of a gas are related. One refinement of the ideal gas law  $pV = RT$  is the **Dieterici equation**  $f(p, V, T) = p(V - b)e^{a/RVT} - RT = 0$ . The constant  $b$  depends on the volume of the molecules and  $a$  depends on the interaction of the molecules. (A different variation of the ideal gas law is van der Waals law (ICE)). Problem: compute  $V_T$ .

If  $V = V(T, p)$ , the chain rule says  $f_V V_T + f_T = 0$ , so that  $V_T = -f_T/f_V = -(-ap(V - b)e^{a/RVT}/(RV^2T^2) - R)/(pVe^{a/RVT} - p(V - b)e^{a/RVT}/(RV^2T))$ . (This could be simplified to  $(R + a/TV)/(RT/(V - b) - a/V^2)$ ).

**MORE GENERAL CHAIN RULE.** (case 2 in the book)

If  $\vec{f}$  is a vector-valued function, we can apply the chain rule for each of the components. The derivative  $\vec{f}'$  of  $\vec{f}$  is then a vector too:  $d/dt f_i(r(t)) = \nabla f_i(r(t)) \cdot r'(t)$ .

**DERIVATIVE.** If  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is a map, its **derivative**  $f'$  is the  $m \times n$  matrix  $[f']_{ij} = \frac{\partial}{\partial x_j} f_i$ .

**EXAMPLES OF DERIVATIVES.**

$f : \mathbf{R} \rightarrow \mathbf{R}^3$ curve $f'$ velocity vector.	$f : \mathbf{R}^3 \rightarrow \mathbf{R}$ scalar function $f'$ gradient vector.	$f : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ surface $f'$ tangent matrix	$f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ coordinate change $f'$ Jacobean matrix.	$f : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ gradient field $f'$ Hessian matrix.
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**THE GENERAL CHAIN RULE.**

If  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  and  $g : \mathbf{R}^k \rightarrow \mathbf{R}^n$ , we can compose  $f \circ g$ , which is a map from  $\mathbf{R}^k$  to  $\mathbf{R}^m$ . The chain rule expresses the derivative of  $f \circ g(x) = f(g(x))$  in terms of the derivatives of  $f$  and  $g$ .

$$\frac{\partial}{\partial x_j} f(g(x))_i = \sum_k \frac{\partial}{\partial x_k} f_i(g(x)) \frac{\partial}{\partial x_j} g_k(x) \quad \text{or short} \quad (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Here  $f'(g(x))$  and  $g'(x)$  are matrices and  $\cdot$  is the matrix multiplication. The chain rule in higher dimensions looks like the chain rule in one dimension, only that the objects are matrices and the multiplication is matrix multiplication.

**EXAMPLE. GRADIENT IN POLAR COORDINATES.** In polar coordinates, the gradient is defined as  $\nabla f = (f_r, f_\theta/r)$ . Using the chain rule, we can relate this to the usual gradient:  $d/dr f(x(r, \theta), y(r, \theta)) = f_x(x, y) \cos(\theta) + f_y(x, y) \sin(\theta)$  and  $d/(rd\theta) f(x(r, \theta), y(r, \theta)) = -f_x(x, y) \sin(\theta) + f_y(x, y) \cos(\theta)$  means that the length of  $\nabla f$  is the same in both coordinate systems.

**PROOFS OF THE CHAIN RULE.**

1. Proof) By linear approximation  $f$  by a function  $L$  it is enough to check the chain rule for linear functions  $f(x, y) = ax + by - c$  and where  $\vec{r}(t) = (x_0 + tu, y_0 + tv)$  is a line. Then  $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt} (a(x_0 + tu) + b(y_0 + tv)) = au + bv$  and this is the dot product of  $\nabla f = (a, b)$  with  $\vec{r}'(t) = (u, v)$ .

2. Proof) Plugging in the definitions of the derivatives using limits.

**WHERE DO WE NEED THE CHAIN RULE (informal).**

While the chain rule is useful in calculations using the composition of functions, the iteration of maps or in doing change of variables, it is also useful for **understanding** some theoretical aspects. Examples:

- In the proof of the fact that **gradients are orthogonal to level surfaces**.
- It appears in **change of variable** formulas.
- It will be used in the **fundamental theorem for line integrals** coming up later in the course.
- The chain rule illustrates also the **Lagrange multiplier** method which we will see later.
- In **fluid dynamics**, PDE's often involve terms  $u_t + u \nabla u$  which give the change of the velocity in the frame of a fluid particle.
- In **chaos theory**, one wants to understand what happens after iterating a map  $f$ .