4/7/2003, 3D INTEGRATION — Math 21a, O. Knill

HOMEWORK: 12.7: 12.7: 4, 16, 24, 34, 46

3D INTEGRATION. If $f(x,y,z)$ is a function of three variables and R is a region in space, then $\int\int\int_R f(x,y,z)\ dxdydz$ is defined as the limit of Riemann sum $\frac{1}{n^3}\sum_{\vec{x}_{ijk}\in R} f(\vec{x}_{ijk})$ for $n \to \infty$, where $\vec{x}_{ijk} = (\frac{i}{n}, \frac{j}{n}, \frac{k}{n})$.

TRIPLE INTEGRALS. As in two dimensions, triple integrals can be evaluated through iterated 1D integrals.

EXAMPLE. If R is the box $[0,1]\times[0,1]\times[0,1]$ and let $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dx\ dy\ dz .$$

CALCULATION. We start from the core $\int_0^1 24x^2y^3z\ dx = 12x^3y^3$, then integrate the middle layer: $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally handle the outer layer: $\int_0^1 3x^2 dx$ =1.

WHAT DID WE DO? When we calculate the most inner integral, we fix z and y. The integral is the average of $f(x,y,z)$ along a line intersected with the body. After completing the second integral, we have computed the average on the plane $z = const$ intersected with R. The most outer integral averages all these two dimensional sections.

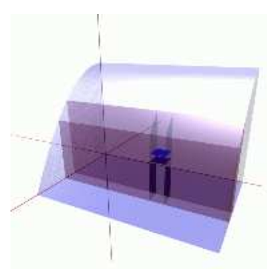
VOLUME OF THE SPHERE. (We will do this more elegantly later). The volume is

$$V = \int\int\int_R dxdydz = \int_{-1}^{1} [\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz]dy]dx$$

After computing the inner integral, we have $V = 2\int_{-1}^1 [\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)^{1/2} dy]dx$.
To resolve the next layer, call $1 - x^2 = a^2$. The task is to find $\int_0^a \sqrt{a^2-x^2}\ dx$. Make the substitution $x/a = \sin(u), dx = a\cos(u)$ to write this as $a\int_0^{\arcsin(a/a)} \sqrt{1-\sin^2(u)} a\cos(u) du = a^2 \int_0^{\pi/2} \cos^2(u) du = a^2\pi/2$. At this stage we have computed the area of a disc of radius a (which is the intersection of the sphere with the plane $x = const$. And we can finished up by calculating the last, the most outer integral

$$V = 2\pi/2 \int_{-1}^{1} (1-x^2)\ dx = 4\pi/3\ .$$

MASS OF A BODY. In general, the mass of a body with density $\rho(x,y,z)$ is $\int\int\int_R \rho(x,y,z)\ dV$. For bodies with constant density ρ the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is 1.



$$\int_0^2 \int_0^6 \int_0^{4-x^2} dz\ dy\ dx = \int_0^2 \int_0^6 (4-x^2)\ dydx$$

$$= \quad 6\int_0^2 (4-x^2)\ dx = 6(4x - x^3/3|_0^2 = 32$$

CENTER OF MASS. Compute the center of mass of the same body. The center of mass is $(24/32, 96/32, 256/180) = (3/4, 3, 8/5)$:

$$\int_0^2 \int_0^6 \int_0^{4-x^2} x \, dz \, dy \, dx = \int_0^2 \int_0^6 x(4-x^2) \, dydx = 6 \int_0^2 x(4-x^2) \, dx = 24x^2/2 - 6x^4/4|_0^2 = 24$$

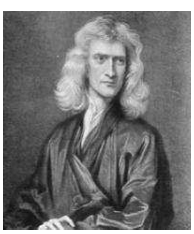
$$\int_0^2 \int_0^6 \int_0^{4-x^2} y \, dz \, dy \, dx = \int_0^2 \int_0^6 y(4-x^2) \, dydx = \int_0^2 18(4-x^2) \, dx = 18(4x - x^3/3)|_0^2 = 96$$

$$\int_0^2 \int_0^6 \int_0^{4-x^2} z \, dz \, dy \, dx = \int_0^2 \int_0^6 (4-x^2)^2/2 \, dydx = 6 \int_0^2 (4-x^2)^2/2 \, dx = 3(16x - 8x^3/3 + x^5/5)|_0^2 = 256/5$$

SOME HISTORY OF COMPUTING VOLUMES. How did people come up calculating the volume $\int \int \int_R 1 \, dxdydz$ of a body?

Archimedes ((-287)-(-212)): Archimedes's method of integration allowed him to find areas, volumes and surface areas in many cases. His method of exhaustion paths the numerical method of integration by Riemann sum. The **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies.

Cavalieri (1598-1647): Cavalieri could determined area and volume using tricks like the **Cavalieri principle**. Example: to get the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$.



Newton (1643-1727) and Leibniz(1646-1716): Newton and Leibniz, developed calculus independently. The new tool made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools.

MONTE CARLO COMPUTATIONS. Here is an other way to compute integrals: Suppose we want to calculate the volume of some body R inside the unit cube $[0,1] \times [0,1] \times [0,1]$. The **Monte Carlo method** is to shoot randomly n times onto the unit cube and count the fraction of times, we hit the solid. Here is an experiment with Mathematica and where the body is one eights of the unit ball:

$$\mathrm{R := Random[]; k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k++], \{10000\}]; k/10000}$$

Assume, we hit 5277 of n=10000 times. The volume so measured is 0.5277. The actual volume of 1/8'th of the sphere is $\pi/6 = 0.524$. For $n \to \infty$ the Monte Carlo computation gives the actual volume.

WHERE CAN TRIPLE INTEGRALS OCCUR?

- Calculation of volumes $V = \int \int \int_R 1 \, dV$, masses $M = \int \int \int \rho \, dV$.
- Finding averages $\int \int \int_R f \, dV / \int \int \int_R 1 \, dV$. Examples: average algae concentration in a swimming pool.
- Determining probabilities. Example: quantum probability.
- Moment of inertia $\int \int \int_R r(x,y,z)^2 \rho(x,y,z) \, dV$, where $r(x,y,z)$ is the distance to the axes of rotation.
- Center of mass $(\int \int \int_R x\rho \, dV/M, \int \int \int_R y\rho \, dV/M, \int \int \int_R z\rho \, dV/M)$.