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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, and 5, we need to see **details** of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

- 1)  T  F Every function  $f(x, y)$  of two variables has either a global minimum or a global maximum.

**Solution:**

Take for example  $f(x, y) = x + y$ . This function has a constant nonzero gradient and so no critical point. It is unbounded above and below.

- 2)  T  F The linearization of the function  $f(x, y) = e^{x+3y}$  at  $(0, 0)$  is  $L(x, y) = 1 + x + 3y$ .

**Solution:**

Use the definition of linearization. The gradient of  $f$  is  $\nabla f = \langle e^{x+3y}, 3e^{x+3y} \rangle$ . At  $(0, 0)$  this is  $\langle 1, 3 \rangle$ . We have  $f(0, 0) = 1$  so that  $L(x, y) = 1 + x + 3y$ .

- 3)  T  F The function  $f(x, y, z) = x^2 \cos(z) + x^3 y^2 z + (y - 2)^3 y^5$  satisfies the partial differential equation  $f_{xyxzxy} = 12$ .

**Solution:**

Use Clairot.

- 4)  T  F If  $xe^z = y^2z$ , then  $\partial z / \partial x = e^z / (y^2 - xe^z)$ .

**Solution:**

This is a direct application of implicit differentiation  $z_x = -f_x / f_z$ .

- 5)  T  F The function  $\cos(x^2) \cos(y^2)$  has a local maximum at  $(0, 0)$ .

**Solution:**

The value at  $(0, 0)$  is equal to 1. The functions and so the product take values between  $-1$  and  $1$ .

- 6)  T  F The value of the double integral  $\int_0^{\pi/4} \int_0^2 x^3 \cos(y) dx dy$  is the same as  $(\int_0^2 x^3 dx)(\int_0^{\pi/4} \cos(y) dy)$ .

**Solution:**

The function  $\cos(y)$  is a constant for the inner integral so that we can pull it out of the inner integral.

- 7)  T  F The gradient of  $f(x, y)$  is always tangent to the level curves of  $f$ .

**Solution:**

It is perpendicular

- 8)  T  F If  $f(x, y, z) = x - 2y + z$ , then the largest possible directional derivative  $D_{\vec{u}}f$  at any point in space is  $\sqrt{6}$ .

**Solution:**

The gradient has length  $\sqrt{6}$ . The directional derivative into the direction of the gradient is the length of the gradient.

- 9)  T  F  $\int_0^1 \int_0^1 (x^2 + y^2) dx dy = \int_0^1 \int_0^1 r^3 dr d\theta$ .

**Solution:**

While the substitution of the function and the  $r$  factor have been done correctly, the region changes. The right integral defines a sector, while the left integral is an integral over the unit square.

- 10)  T  F It is possible that the directional derivative  $D_{\vec{v}}f$  is positive for all unit vectors  $\vec{v}$ .

**Solution:**

The directional derivative changes sign if  $\vec{v}$  is replaced by  $-\vec{v}$ .

- 11)  T  F Using linearization of  $f(x, y) = xy$  we can estimate  $f(0.999, 1.01) \sim 1 - 0.001 + 0.01 = 1.009$ .

**Solution:**

$L(x, y) = 1 - 1 \cdot 0.001 + 1 \cdot 0.01$ .

- 12)  T  F Given a curve  $\vec{r}(t)$  on a surface  $g(x, y, z) = -1$ , then  $\frac{d}{dt}g(\vec{r}(t)) < 0$ .

**Solution:**

It is zero.

- 13)  T  F If  $f(x, y)$  has a local minimum at  $(0, 0)$  then it is possible that  $f_{xy}(0, 0) > 0$ .

**Solution:**

$D = f_{xx}f_{yy} - f_{xy}^2 > 0$  is still possible, if  $f_{xx}$  and  $f_{yy}$  are large. For example  $x^2 + y^2 + xy/10$  has a local minimum at  $(0, 0)$  even so  $f_{xy} > 0$ .

- 14)  T  F The function  $f(x, y) = -x^8 - 2x^6 - y^8$  has a local minimum at  $(0, 0)$ .

**Solution:**

One can not use the second derivative test because the discriminant is zero. But the function is zero at  $(0, 0)$  and strictly negative everywhere else. Therefore,  $(0, 0)$  is a global maximum. It is definitely not a minimum.

- 15)  T  F If  $\vec{r}(t)$  is a curve in space and  $f$  is a function of three variables, then  $\frac{d}{dt}f(\vec{r}(t)) = 0$  for  $t = 0$  implies that  $\vec{r}(0)$  is a critical point of  $f(x, y, z)$ .

**Solution:**

We can have  $r(t) = (t, 0, 0)$  and  $f(x, y, z) = x^2 + (y - 1)^2$ .

- 16)  T  F Let  $a, b, c$  be the number of saddle points, maxima and minima of a function  $f(x, y)$ . Then  $a \leq b + c$ .

**Solution:**

Already  $x^2 - y^2$  is a counter example.

- 17)  T  F If  $f(x, y)$  is a nonzero function of two variables and  $R$  is a region, then  $\int \int_R f(x, y) dx dy$  is the volume under the graph of  $f$  and therefore a positive value.

**Solution:**

if  $f$  is replaced by  $-f$ , then the sign of the integral changes too.

- 18)  T  F We extremize  $f(x, y)$  under the constraint  $g(x, y) = c$  and obtain a solution  $(x_0, y_0)$ . If the Lagrange multiplier  $\lambda$  is positive, then the solution is a minimum.

**Solution:**

There is no relation between the sign of  $\lambda$  and minima and maxima. Change  $g = c$  to  $-g = -c$  and the sign of  $\lambda$  changes.

- 19)  T  F The tangent plane to a surface  $f(x, y, z) = 1$  intersects the surface in exactly one point.

**Solution:**

Take a one sheeted hyperboloid.

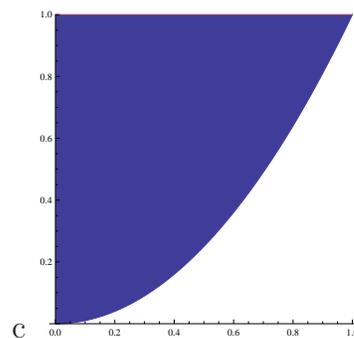
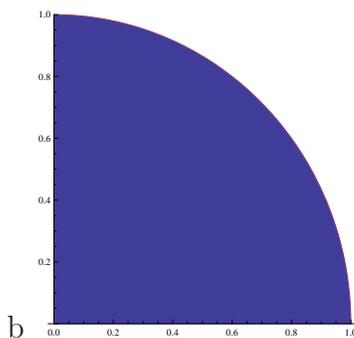
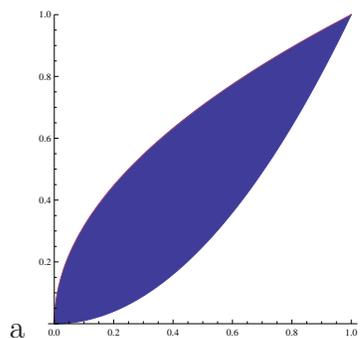
- 20)  T  F Let  $\vec{v}$  be a vector of length 1 in space. Given a function  $f(x, y, z)$  of three variables. If  $(x_0, y_0, z_0)$  is a critical point of  $f$ , then it is a critical point of  $g(x, y, z) = D_{\vec{v}}f(x, y, z)$ .

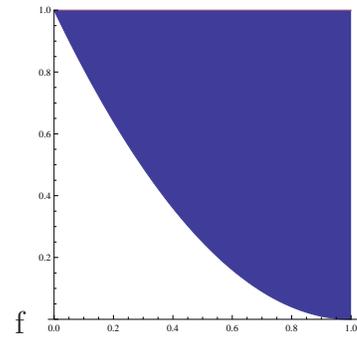
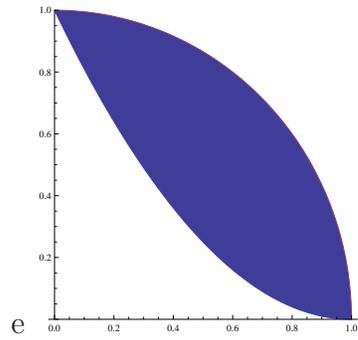
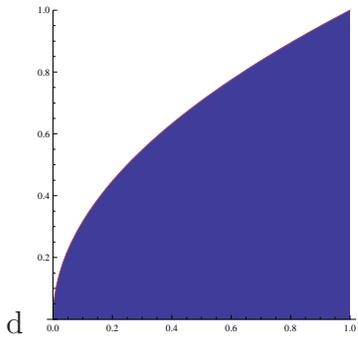
**Solution:**

Let  $\vec{v} = \langle 1, 0, 0 \rangle$ . Now  $g(x, y, z) = f_x(x, y, z)$  and  $\nabla g = \langle f_{xx}, f_{xy}, f_{xz} \rangle$ .

Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding double integrals





Enter a,b,c,d,e or f	Integral of $f(x, y)$	Enter a,b,c,d,e or f	Integral of $f(x, y)$
	$\int_0^1 \int_{x^2}^{\sqrt{x}} f(x, y) dy dx$		$\int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y) dy dx$
	$\int_0^1 \int_0^{\sqrt{y}} f(x, y) dx dy$		$\int_0^1 \int_{(1-x)^2}^1 f(x, y) dy dx$
	$\int_0^1 \int_{y^2}^1 f(x, y) dx dy$		$\int_0^1 \int_{(1-x)^2}^{\sqrt{1-x^2}} f(x, y) dy dx$

b) (4 points) Match the PDE's with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$f_{xx} = -f_{yy}$
	$f_x = f_y$

Enter A,B,C,D here	PDE
	$f_{xx} = f_{yy}$
	$f_x = f_{yy}$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation
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**Solution:**

- a) b  
 c) f  
 d) e  
 b) D A  
 C B

Problem 3) (10 points)

- a) (3 points) Find and classify all the critical points of  $f(x, y) = xy - x$  on the plane.
- b) (2 points) Decide whether an absolute maximum or an absolute minimum of  $f$  exists on the plane  $\mathbb{R}^2$ .
- c) (3 points) Use the method of Lagrange multipliers to find the maximum and minimum of  $f$  on the boundary  $x^2 + 4y^2 = 12$  of the elliptical region  $G : x^2 + 4y^2 \leq 12$ .
- d) (2 points) Find the absolute maximum and absolute minimum of  $f$  on the region  $G$  given in c).

**Solution:**

a)  $\nabla f = \langle y - 1, x \rangle = \vec{0}$  for  $(x, y) = (0, 1)$ . Since  $f_{xx} = f_{yy} = 0$  and  $f_{xy} = 1$ , the discriminant is  $D = 0^2 - 2^2 < 0$  and  $(0, 1)$  is a saddle point.

b) There is no global maximum, nor any global minimum on the plane. On the  $x$ -axis  $y = 0$  for example, we have  $f(x, 0) = -x$  which is unbounded both from above and from below.

c) The Lagrange equations are

$$\begin{aligned} y - 1 &= \lambda 2x \\ x &= \lambda 8y \\ x^2 + 4y^2 &= 12. \end{aligned}$$

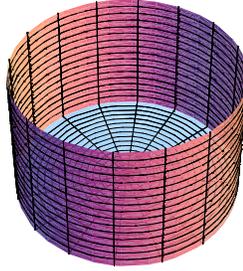
$y \neq 0$ , because otherwise the second equation would give  $x = 0$ , contradicting the constraint. Also  $x \neq 0$ , because otherwise, the first equation would give  $y = 1$ , again contradicting the constraint. Dividing the first by the second gives  $(y-1)/x = (1/4)x/y$  or  $4y(y-1) = x^2$ . Plugging this into the constraint gives  $4y(y-1) + 4y^2 = 12$ . The solutions of this quadratic equation are  $y = 3/2$  or  $y = -1$ . The extrema are  $(\pm 2\sqrt{2}, -1)$  and  $(\pm\sqrt{3}, 1.5)$ .

Since  $f(2\sqrt{2}, -1) = -4\sqrt{2}$ ,  $f(-2\sqrt{2}, -1) = 4\sqrt{2}$ ,  $f(\sqrt{3}, 1.5) = \frac{\sqrt{3}}{2}$  and  $f(-\sqrt{3}, 1.5) = -\frac{\sqrt{3}}{2}$ , the maximum is  $(x, y) = (-2\sqrt{2}, -1)$  and the minimum is  $(x, y) = (2\sqrt{2}, -1)$ .

d) From parts (a) and (c) we have a list of all candidates for global extrema. The global maximum value of  $f$  on  $G$  is  $f(-2\sqrt{2}, -1) = 4\sqrt{2}$ , the global minimal value on  $G$  is  $f(2\sqrt{2}, -1) = -4\sqrt{2}$ .

Problem 4) (10 points)

Find the cylindrical basket which is open on the top has the largest volume for fixed area  $\pi$ . If  $x$  is the radius and  $y$  is the height, we have to extremize  $f(x, y) = \pi x^2 y$  under the constraint  $g(x, y) = 2\pi xy + \pi x^2 = \pi$ . Use the method of Lagrange multipliers.



**Solution:**

The Lagrange equations are

$$\begin{aligned} 2xy\pi &= (2x\pi + 2y\pi)\lambda \\ \pi x^2 &= 2\pi x\lambda \\ \pi x^2 + 2\pi xy &= \pi \end{aligned}$$

Since  $x = 0$  is not possible (it would violate the constraint), we can divide the second equations by  $x$  and divide the first by the second equation. This gives  $x = y = 1/\sqrt{3}$ .

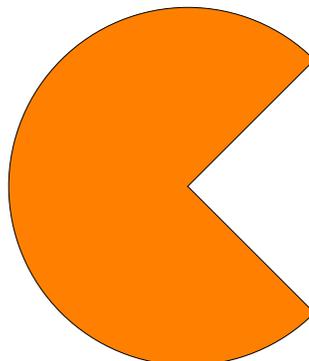
The maximum value is  $\pi\sqrt{3}/9$ .

Problem 5) (10 points)

The Pac-Man region  $R$  is bounded by the lines  $y = x, y = -x$  and the unit circle. The number

$$a = \frac{\int \int_R x \, dx dy}{\int \int_R 1 \, dx dy}$$

defines the point  $C = (a, 0)$  called center of mass of the region. Find it.



**Solution:**

$$\int_{\pi/4}^{7\pi/4} \int_0^1 r \cos(\theta) r dr d\theta = (1/3) \sin(\theta) \Big|_{\pi/4}^{7\pi/4} = -\sqrt{2}/3 .$$

$$\int_{\pi/4}^{7\pi/4} \int_0^1 r dr d\theta = (1/2)(7\pi/4 - \pi/4) = 6\pi/8 = 3\pi/4 .$$

The second integral is the area of the Pac-Man, which is  $3/4$  of the area of the full disc. Dividing the first by the second integral gives the result  $a = -4\sqrt{2}/(9\pi)$ . The center of mass is  $(-4\sqrt{2}/(9\pi), 0)$ .

Problem 6) (10 points)

a) (5 points) Find the tangent plane to the surface  $\sqrt{xyz} = 60$  at  $(x, y, z) = (100, 36, 1)$ .

b) (5 points) Estimate  $\sqrt{100.1 * 36.1 * 0.999}$  using linear approximation. Here, for clarity reasons, we use \* for the usual multiplication for numbers.

**Solution:**

a) We have

$$\nabla f(x, y, z) = \left\langle \sqrt{\frac{yz}{x}}, \sqrt{\frac{xz}{y}}, \sqrt{\frac{xy}{z}} \right\rangle / 2$$

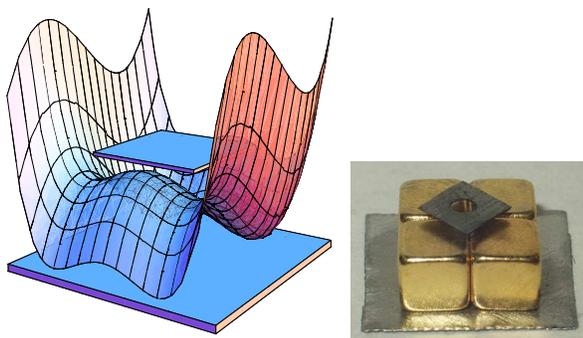
$$\nabla f(100, 36, 1) = \left\langle \frac{6}{10}, \frac{10}{6}, 60 \right\rangle / 2$$

The tangent plane is  $(3/10)x + (5/6)y + 30z = 90$ . We have obtained the constant on the right by plugging in the point  $(x, y, z) = (100, 36, 1)$ .

b) Since  $f(100, 36, 1) = 60$ , we have  $L(x, y, z) = 60 + (3/10)(x-100) + (5/6)(y-36) + 30(z-1)$ . We have  $L(100.1, 36.1, 0.999) = 60 + 0.03 + 0.08333... - 0.03 = 60.08333... = 60 + 1/12$ . This is very close to the actual value  $60.0832455...$ . You have in this problem computed the square root of a real number by hand with an accuracy of 4 digits after the comma.

Problem 7) (10 points)

Oliver got a diammagnetic kit, where strong magnets produce a force field in which pyrolytic graphic flots. The gravitational field produces a well of the form  $f(x, y) = x^4 + y^3 - 2x^2 - 3y$ . Find all critical points of this function and classify them. Is there a global minimum?



Right picture credit: Wikipedia.

**Solution:**

To find the critical points, we have to solve the system of equations  $f_x = 4x^3 - 4x = 0$ ,  $f_y = 3y^2 - 3 = 0$ . The first equation gives  $x = 0$  or  $x = \pm 1$ . The second equation  $f_y = 3y^2 - 3 = 0$  gives  $y = \pm 1$ . There are  $3 \cdot 2 = 6$  critical points. We compute the discriminant  $D = 6y(12x^2 - 4)$  and  $f_{xx} = 12x^2 - 4$  at each of the 6 points and use the second derivative test to determine the nature of the critical point.

point	$D$	$f_{xx}$	nature	value
$(-1, -1)$	-48	8	saddle	1
$(-1, 1)$	48	8	min	-3
$(0, -1)$	24	-4	max	2
$(0, 1)$	-24	-4	saddle	-2
$(1, -1)$	-48	8	saddle	1
$(1, 1)$	48	8	min	-3

There is no global minimum, nor any global maximum since for  $x = 0$ , the function is  $f(0, y) = y^3 - 3y$  which is unbounded from above and from below (it goes to  $\pm\infty$  for  $y \rightarrow \pm\infty$ ).

Problem 8) (10 points)

Let  $f(x, y) = xy$ .

- a) (2 points) Find the direction of maximal increase at the point  $(1, 1)$ .
- b) (3 points) Find the directional derivative at  $(1, 1)$  in the direction  $\langle 3/5, 4/5 \rangle$ .
- c) (2 points) The curve  $\vec{r}(t) = \langle \sqrt{2} \sin(t), \sqrt{2} \cos(t) \rangle$  passes through the point  $(1, 1)$  at some time  $t_0$ . Find  $\frac{d}{dt} f(\vec{r}(t))$  at time  $t_0$  directly.
- d) (3 points) Find  $\frac{d}{dt} f(\vec{r}(t))$  at time  $t_0$  using the multivariable chain rule.

**Solution:**

a)  $\nabla f(x, y) = \langle y, x \rangle$ ,  $\nabla f(1, 1) = \langle 1, 1 \rangle$ . The direction of maximal increase is  $\boxed{\langle 1, 1 \rangle / \sqrt{2}}$ .

b)  $D_v f(1, 1) = \langle 1, 1 \rangle \cdot \langle 3/5, 4/5 \rangle = \boxed{7/5}$ .

c) It is at the time  $t_0 = \pi/4$ , where the curve passes through the point  $(1, 1)$ . We have  $f(\vec{r}(t)) = 2 \cos(t) \sin(t) = \sin(2t)$  and  $d/dt f(\vec{r}(t)) = 2 \cos(2t)$  which is  $\boxed{0}$  at time  $t = \pi/4$ .

d) By the multi variable chain rule,  $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(1, 1) \cdot \langle -\sin(\pi/4), \cos(\pi/4) \rangle = \boxed{0}$ .

**Problem 9) (10 points)**

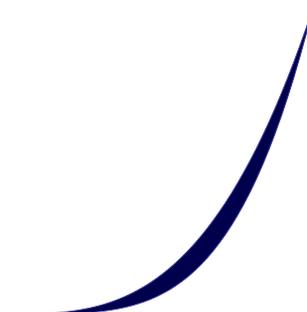
Integrate the function

$$f(x, y) = \frac{y^5 - 1}{y^{1/3} - y^{1/4}}$$

on the finite region bounded by the curves  $y = x^3$  and  $y = x^4$ .

**Solution:**

Make a picture! The two graphs intersect at 0 and 1 forming a grass shaped region.



The type I integral

$$\int_0^1 \int_{x^4}^{x^3} \frac{y^5 - 1}{y^{1/3} - y^{1/4}} dy dx$$

can not be evaluated (at least not without going through difficult substitution/partial fraction procedures which can fill pages).

We decide therefore, to change the order of integration and write a type II integral:

$$\int_0^1 \int_{y^{1/3}}^{y^{1/4}} \frac{y^5 - 1}{y^{1/3} - y^{1/4}} dx dy$$

Now the inner integral can be solved and give  $(1 - y^5)$ . We end up with  $\int_0^1 (1 - y^5) dy =$

$\boxed{5/6}$ .

Problem 10) (10 points)

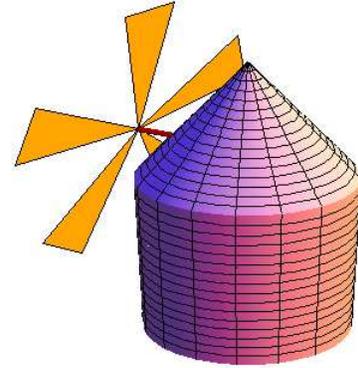
The main building of a mill has a cone shaped roof and cylindrical walls. If the cylinder has radius  $r$ , the height of the side wall is  $h$  and the height of the roof is  $h$ , then the volume is

$$V(h, r) = \pi r^2 h + h\pi r^2/3 = (4\pi/3)hr^2$$

and assume the cost of the building is

$$A(h, r) = \pi r^2 + 2\pi r h + \pi 2r^2 = \pi(3r^2 + 2rh)$$

which is the area of the ground plus the area of the wall plus  $2\pi r h$ , the cost for the roof. For fixed volume  $V(h, r) = 4\pi/3$ , minimize the cost  $A(h, r)$  using the Lagrange multiplier method.



**Solution:**

After dividing out some constants and taking  $g = hr^2 = 1$ , the Lagrange equations become

$$\begin{aligned} 6r + 2h &= \lambda 2hr \\ 2r &= \lambda r^2 \\ r^2 h &= 1 \end{aligned}$$

The second equation can be divided by  $r$  since  $r = 0$  is incompatible with the third equation. The first can be divided by 2. We get

$$\begin{aligned} 3 * r + h &= \lambda hr \\ 2 &= \lambda r \\ r^2 h &= 1 \end{aligned}$$

You can plug in  $\lambda r$  from the second equation into the first to get

$$\begin{aligned} 3r + h &= 2h \\ r^2 h &= 1 \end{aligned}$$

The first equation shows  $h = 3r$  and plugging this into the third equation gives  $r = 1/3^{1/3}$

and  $h = 3r = 3^{2/3}$ .