

Name:

MWF 9 Chao Li
MWF 9 Thanos Papaïoannou
MWF 10 Emily Riehl
MWF 10 Jameel Al-Aidroos
MWF 11 Oliver Knill
MWF 11 Tatyana Kobylatskaya
MWF 12 Tatyana Kobylatskaya
MWF 12 Yu-Jong Tzeng
TTH 10 Junecue Suh
TTH 10 Pei-Yu Tsai
TTH 11:30 Paul Bourgade

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

- 1)

T	F
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 There are two unit vectors $\vec{v}$, $\vec{w}$ for which the sum $\vec{v} + \vec{w}$ has length $1/3$.
- 2)

T	F
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 For any three vectors, we have $|(\vec{u} \times \vec{v}) \times \vec{w}| = |(\vec{v} \times \vec{w}) \times \vec{u}|$.
- 3)

T	F
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 Denote by $d(P, L)$ the distance from a point P to a line L in space. For any point P and any two lines L, K in space, we have $d(P, L) + d(P, K) \geq d(L, K)$.
- 4)

T	F
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 For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the relation $|\vec{u} \times (\vec{v} \times \vec{w})| \leq |\vec{u}||\vec{v}||\vec{w}|$ holds.
- 5)

T	F
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 Assume a curve $\vec{r}(t)$ has constant speed 1 and constant curvature 1 everywhere. Then the curve $\vec{r}(2t)$ has constant speed 2 and constant curvature $1/2$ everywhere.
- 6)

T	F
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 If the curvature of a space curve is constant 1 and the speed $|\vec{r}'(t)| = 1$ everywhere, then the acceleration satisfies $|\vec{r}''(t)| = 1$ everywhere.
- 7)

T	F
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 If a vector field $\vec{F} = \langle P, Q \rangle$ has $\text{curl}(\vec{F}) = Q_x - P_y = 0$ everywhere and divergence $\text{div}(\vec{F}) = P_x + Q_y = 0$ everywhere, then the vector field must be constant.
- 8)

T	F
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 If the level curve $f(x, y) = 1$ contains both the lines $x = y$ and $x = -y$, then $(0, 0)$ must be a critical point for which $D < 0$.
- 9)

T	F
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 The surface $\vec{r}(u, v) = \langle u^3 \cos(v), u^3 \sin(v), u^3 \rangle$ with $v \in [0, 2\pi)$ and $-\infty \leq u \leq \infty$ is a double cone.
- 10)

T	F
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 There is a non-constant function $f(x, y, z)$ of three variables such that $\text{div}(\text{grad}(f)) = f$.
- 11)

T	F
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 If $\text{curl}(\vec{F}) = \vec{F}$, then the vector field $\vec{F}$ satisfies $\text{div}(\vec{F}) = 0$ everywhere.
- 12)

T	F
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 The equation $\phi = \pi/4$ in spherical coordinates defines a half plane.
- 13)

T	F
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 The tangent plane of $x^3 + y^2 + z^4 = 9$ at $(0, 3, 0)$ is $y = 3$.
- 14)

T	F
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 Assume (x_0, y_0) is not a critical point of $f(x, y)$. It is possible that f increases at (x_0, y_0) most rapidly in the direction $\langle 1, 0 \rangle$ and decreases most rapidly in the direction $\langle 4/5, -3/5 \rangle$.
- 15)

T	F
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 Assume $\vec{F}(x, y, z)$ is defined everywhere except on the z -axis and satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere except on the z -axis, then $\int_C \vec{F} \cdot d\vec{r} = 0$ for all curves C .
- 16)

T	F
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 A point (x_0, y_0) is an extremum of $f(x, y)$ under the constraint $g(x, y) = 0$. If $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then (x_0, y_0) can not be a local maximum on the constraint curve.
- 17)

T	F
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 The vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ can be the curl of another vector field $\vec{G}$.
- 18)

T	F
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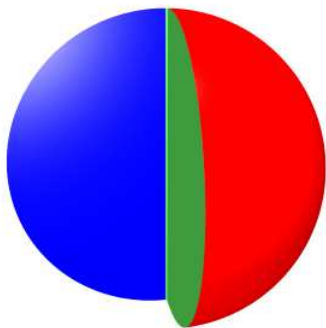
 If $f(x, y)$ and $g(x, y)$ are two functions and $(2, 3, 3)$ is a critical point of the function $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$, then $(2, 3)$ is a solution of the Lagrange equations for extremizing $f(x, y)$ under the constraint $g(x, y) = 0$.
- 19)

T	F
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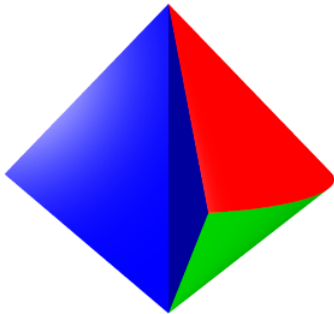
 Assume $(0, 0)$ is a global maximum of $f(x, y)$ on the disc $D = \{x^2 + y^2 \leq 1\}$, then $\int \int_D f(x, y) \, dx dy \leq \pi f(0, 0)$.

- 20) ☐ T ☐ F
- Let C be a curve parametrized by $\vec{r}(t), 0 \leq t \leq 1$ so that its speed is constant
1. Then $\int_C \nabla f \cdot d\vec{r}$ is equal to $\int_0^1 D_{\vec{r}'(t)}(f(\vec{r}(t))) dt$.

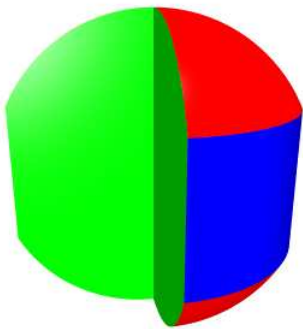
a) (4 points) Match the following triple integrals with the regions.



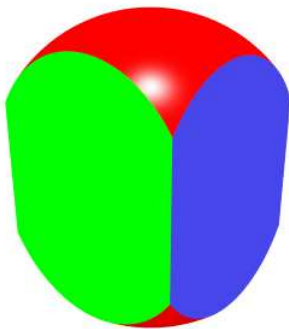
I



II



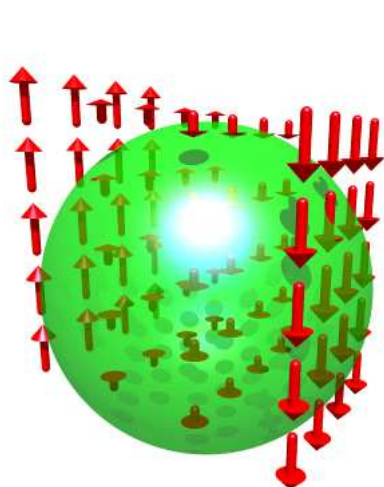
III



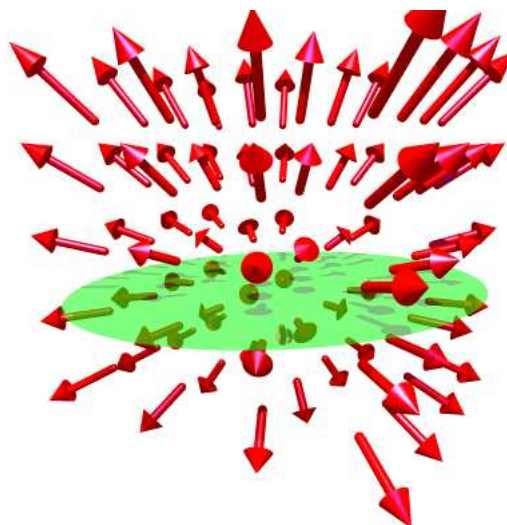
IV

Enter I,II,III,IV here	Equation
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} f(r \cos(\theta), r \sin(\theta), z) r \, dz dr d\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{r-1}^{1-r} f(r \cos(\theta), r \sin(\theta), z) r \, dz dr d\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} f(r \cos(\theta), r \sin(\theta), z) r \, dz dr d\theta$
	$\int_{-1}^1 \int_{-1}^1 \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(x, y, z) \, dz dy dx$

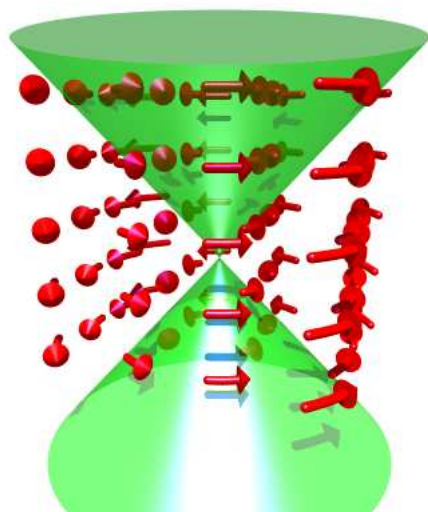
2b) (6 points) Match the following pictures with their vector fields and surfaces. Then check whether the flux integral is zero.



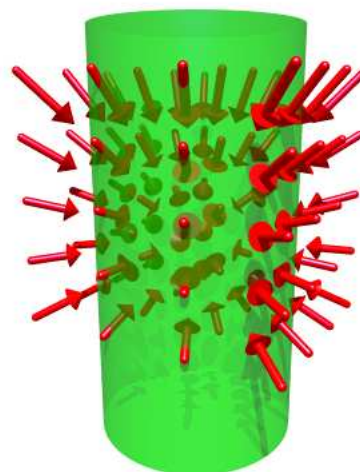
A



B



C

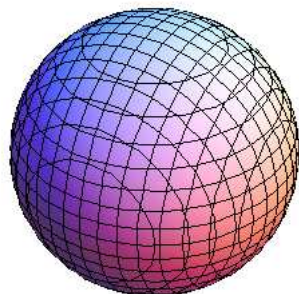


D

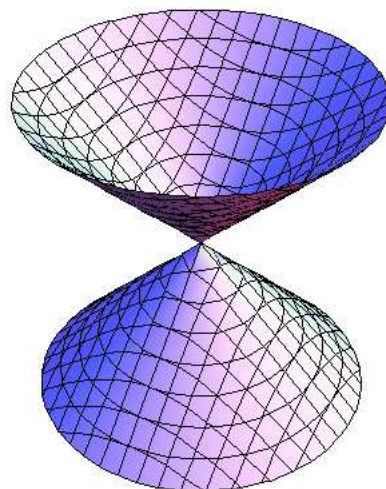
Enter A,B,C,D	Field	Surface	Flux zero
	$\vec{F}(x, y, z) = \langle x, y, z \rangle$	$\vec{r}(u, v) = \langle u, v, 0 \rangle$	
	$\vec{F}(x, y, z) = \langle 0, 0, y \rangle$	$\vec{r}(\theta, \phi) = \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi) \rangle$	
	$\vec{F}(x, y, z) = \langle -x, -y, -z \rangle$	$\vec{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z \rangle$	
	$\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$	$\vec{r}(\theta, z) = \langle z \cos(\theta), z \sin(\theta), z \rangle$	

Problem 3) (10 points)

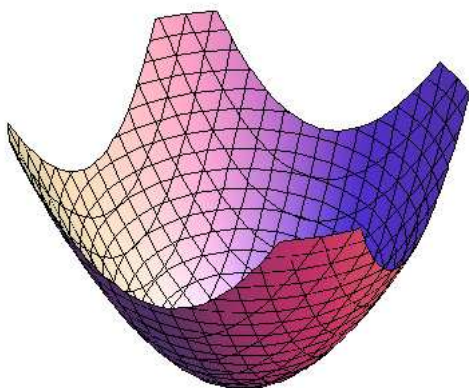
a) (6 points) Match the following level surfaces with functions $f(x, y, z)$ and also match the parametrization of part of the surface $f(x, y, z) = 0$.



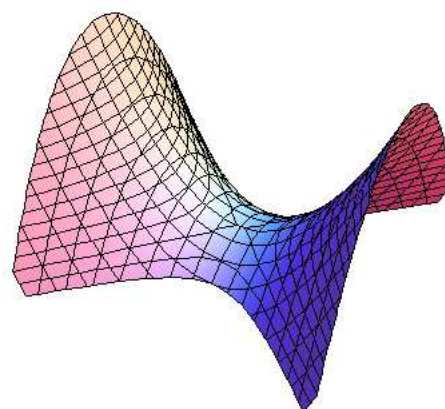
I



II



III



IV

Enter I,II,III,IV	$f(x, y, z) = 0$	Enter I,II,III,IV	parametrization
	$f(x, y, z) = -x^2 + y^2 + z$		$\langle u, v, u^2 - v^2 \rangle$
	$f(x, y, z) = x^2 + y^2 + z^2 - 1$		$\langle u, v, u^2 + v^2 \rangle$
	$f(x, y, z) = -x^2 - y^2 + z$		$\langle u, v, \sqrt{1 - u^2 - v^2} \rangle$
	$f(x, y, z) = -x^2 - y^2 + z^2$		$\langle s \cos(t), s \sin(t), s \rangle$

b) (2 points) We know that
 $\vec{r}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$,
 $\vec{r}(0) = \langle 2, 3, 4 \rangle$ and
 $\vec{r}'(0) = \langle 0, 1, 1 \rangle$.
 The expression $\langle \cos(t)+2, \sin(t)+3, t+4 \rangle$ is equal to:

Check which applies	result
☐	the velocity $\vec{r}'(t)$
☐	the position $\vec{r}(t)$
☐	the curvature $\kappa(\vec{r}(t))$
☐	the unit tangent vector $\vec{T}(t)$

c) (2 points) What is the name of the partial differential equation $\text{div}(\text{grad}(f)) = 0$ for $f(x, y)$?

Check which applies	PDE
☐	Transport equation
☐	Wave equation
☐	Heat equation
☐	Laplace equation

Problem 4) (10 points)

Find the distance between the sphere $(x-4)^2 + y^2 + (z-6)^2 = 1$ and the cylinder of radius 2 around the line $x = y = z$.

Problem 5) (10 points)

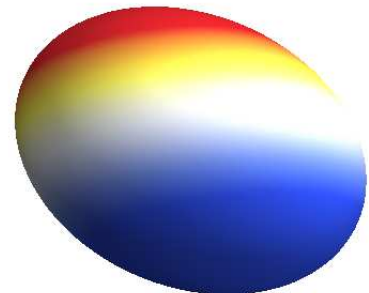
a) (3 points) Find the tangent plane to the surface $S : 4xy - z^2 = 0$ at $(1, 1, 2)$.

b) (4 points) Estimate $4 * 1.001 * 0.99 - 2.001^2$, where $*$ is the usual multiplication.

c) (3 points) Parametrize the line through $(1, 1, 2)$ which is perpendicular to the surface S at $(1, 1, 2)$.

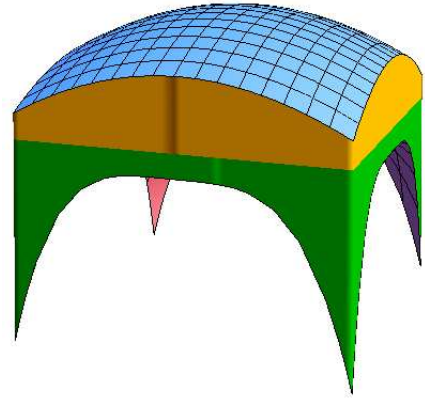
Problem 6) (10 points)

Find the place on the elliptical **asteroid** surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$, where the temperature $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximal.



Problem 7) (10 points)

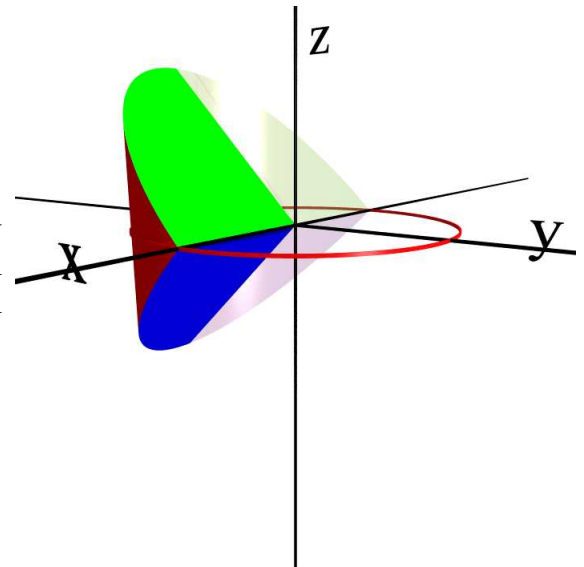
The thickness of the region enclosed by the two graphs $f_1(x, y) = 10 - 2x^2 - 2y^2$ and $f_2(x, y) = -x^4 - y^4 - 2$ is denoted by $f(x, y) = f_1(x, y) - f_2(x, y)$. Classify all critical points of f and find the global minimal thickness.



To the picture: over a square domain, the region looks like a chair. In the problem you consider the function over the entire plane.

Problem 8) (10 points)

Find the volume of the solid piece of **cheese** bound by the cylinder $x^2 + y^2 = 1$, the planes $y - z = 0$ (bottom boundary) and $y + z = 0$ (top boundary) which is on the quadrant $x \geq 0$ and $y \leq 0$.



Problem 9) (10 points)

Compute the surface area of the **Tsai** surface which is parametrized by

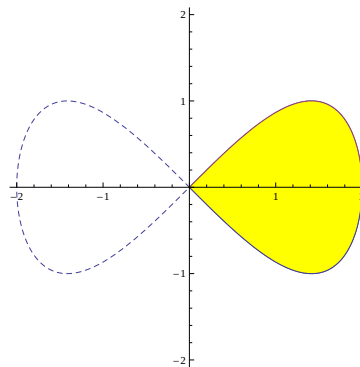
$$\vec{r}(u, v) = \langle 3u + 2v, 4u + v, \frac{2}{7}v^{\frac{7}{2}} \rangle,$$

where $0 \leq u \leq 1$ and $u^{1/4} \leq v \leq 1$.

Problem 10) (10 points)

Find the area $\int \int_R 1 \, dx dy$ of the region R inside the right leaf of the **Geronon lemniscate** $x^4 = 4(x^2 - y^2)$ which has the parametrization

$$\vec{r}(t) = \langle 2 \sin(t), 2 \sin(t) \cos(t) \rangle .$$



Problem 11) (10 points)

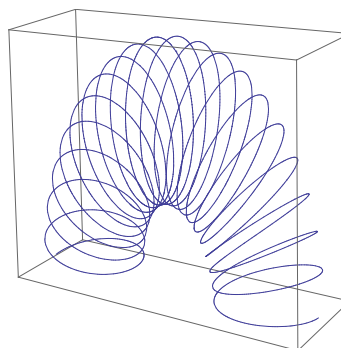
Find the line integral of the vector field

$$\vec{F}(x, y, z) = \langle \cos(x + z), 2yz e^{y^2 z} + 7, \cos(x + z) + y^2 e^{y^2 z} \rangle$$

along the **slinky** curve

$$\vec{r}(t) = \langle \sin(40t), (2 + \cos(40t)) \cos(t), (2 + \cos(40t)) \sin(t) \rangle$$

with $0 \leq t \leq \pi$.



Problem 12) (10 points)

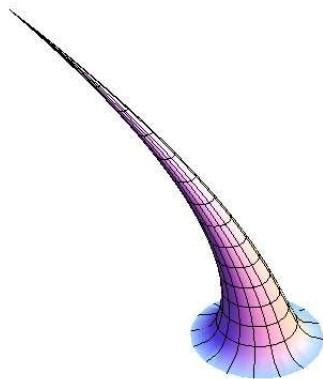
Find the flux integral $\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x, y, z) = \langle 2 \cos(\pi y) e^{2x} + z^2, x^2 \cos(z\pi/2) - \pi \sin(\pi y) e^{2x}, 2xz \rangle$$

and S is the **thorn** surface parametrized by

$$\vec{r}(s, t) = \langle (1 - s^{1/3}) \cos(t) - 4s^2, (1 - s^{1/3}) \sin(t), 5s \rangle$$

with $0 \leq t \leq 2\pi, 0 \leq s \leq 1$ and oriented so that the normal vectors point to the outside of the thorn.

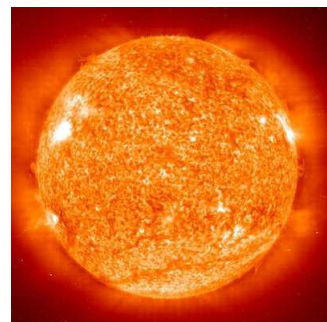


Problem 13) (10 points)

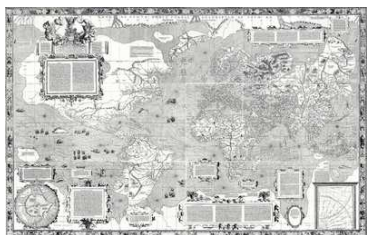
Assume the vector field

$$\vec{F}(x, y, z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int_S \vec{F} \cdot d\vec{S}$.



Problem 14) (10 points)



The **Mercator projection** is one of the most famous map projections. It was invented in 1569 and used for nautical voyages. The inverse of the projection is the parametrization of the sphere as

$$\vec{r}(u, v) = \langle \cos(u) \cos(\arctan(\sinh(v))), \sin(u) \cos(\arctan(\sinh(v))), \sin(\arctan(\sinh(v))) \rangle .$$

- (3 points) Show that $|\vec{r}(u, v)| = 1$ verifying so that $\vec{r}(u, v)$ parametrizes the unit sphere, if $0 \leq u < 2\pi, -\infty < v < \infty$.
- (3 points) Show that $|\vec{r}_u(u, v)| = |\vec{r}_v(u, v)| = 1/\cosh(v)$ and that $\vec{r}_u(u, v) \cdot \vec{r}_v(u, v) = 0$.
- (2 points) Use *b)* to show that $|\vec{r}_u \times \vec{r}_v| = 1/\cosh(x)^2$.
- (2 points) Use $\int 1/\cosh^2(x) dx = 2 \arctan(\tanh(x/2)) + C$ to see that the surface area of the unit sphere is 4π .

Hint for b): you can use the identity $\cos(\arctan(\sinh(v))) = 1/\cosh(v)$.