11/1/2011 SECOND HOURLY PRACTICE VI **Math 21a, Fall 2011**

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for multiple choice problems, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T F Given a unit vector v, define $g(x) = D_v f(x)$. If at a critical point, for all vectors v we have $D_v g(x) > 0$, then f is a local maximum.

2) T F Assume f satisfies the PDE $f_x = f_y$. If $g = f_x$, then $g_x = g_y$.

3) T F The equation $\phi = \pi/4$ in spherical coordinates ($\rho \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ as usual) and the surface $x^2 + y^2 = z^2$ (with no further restrictions on x, y, z) are the same surface.

4) T F Even with $f_x(a, b) = 0$ and $f_y(a, b) = 0$, it is possible that some directional derivative $D_{\vec{v}}(f)$ of $f(x, y)$ at (a, b) is non-zero.

5) T F There exists a pair of different points on a sphere, for which the tangent planes are parallel.

6) T F If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then $D_u f(x, y, z) = 0$.

7) T F Assume we have a smooth function $f(x, y)$ for which the lines $x = 0, y = 0$ and $x = y$ are level curves $f(x, y) = 0$. Then $(0, 0)$ is a critical point with $D < 0$.

8) T F The gradient of $f(x, y)$ is perpendicular to the graph of f.

9) T F The level curves of a linearization $L(x, y)$ of a function $f(x, y) = \sin(x + y)$ at $(0, 0)$ consist of lines.

10) T F If $x^4 y + \sin(y) = 0$ then $y' = 4x^3 y/(x^4 + \cos(y))$.

11) T F The linearization $L(x, y)$ at a critical point (x_0, y_0) of a function $f(x, y)$ is a constant function.

12) T F The surface $x^2 + y^2 - z^2 = 1$ has a parametrization of the form $\langle x(s, t), y(s, t), z(s, t) \rangle = \langle s, t, f(s, t) \rangle$ for some function $f(s, t)$ for which the parametrization covers the entire surface.

13) T F The tangent plane to the graph of $f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$ is a level surface of the linearization $L(x, y, z)$ of $z - f(x, y)$.

14) T F The critical points of $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ are solutions to the Lagrange equations when extremizing the function $f(x, y)$ under the constraint $g(x, y) = 0$.

15) T F The curve defined by $z = 1, \theta = \frac{\pi}{4}$ in cylindrical coordinates is a circle.

16) T F If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

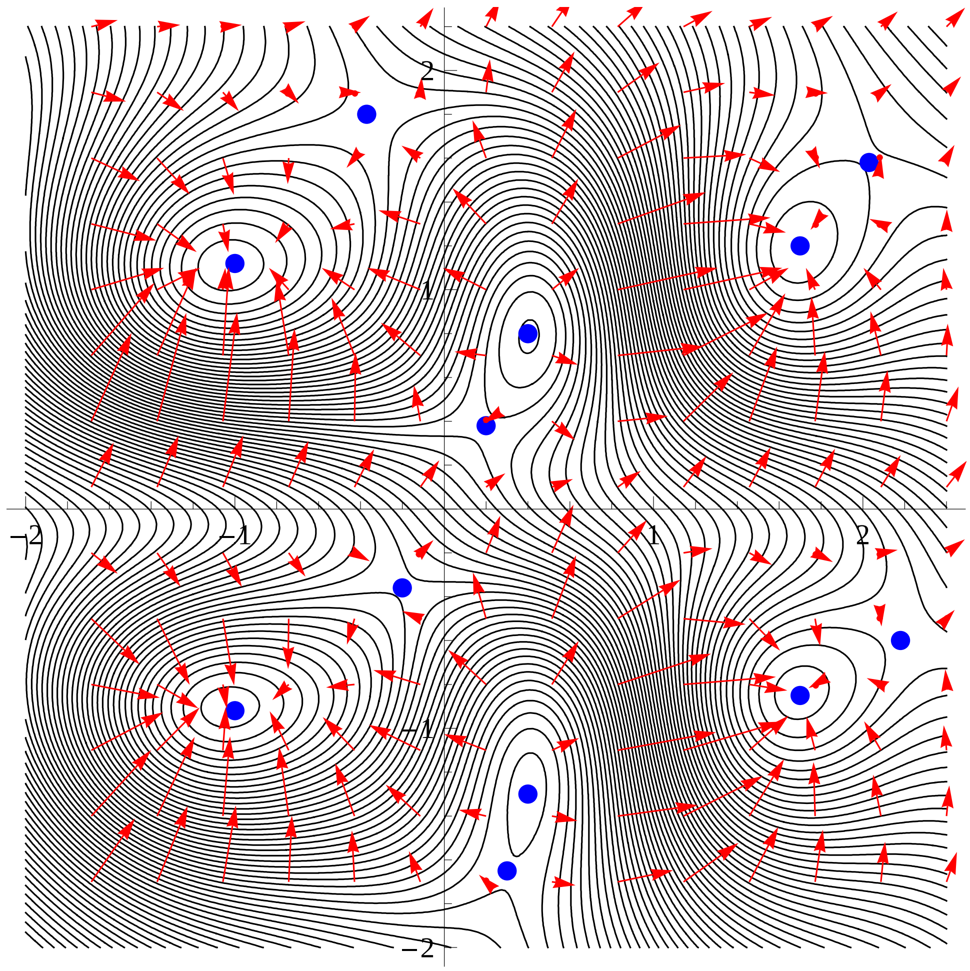
17) T F If $f(x, y, z) = x^2 + y^2 + z^2$, then $\nabla f = 2x + 2y + 2z$.

18) T F A function $f(x, y)$ in the plane always has a local minimum or a local maximum.

19) T F For any smooth function $f(x, y)$, the inequality $\|\nabla f\| \geq |f_x + f_y|$ is true.

20) T F If a function $f(x, y)$ satisfies $|\nabla f(x, y)| = 1$ everywhere in the plane, then f is constant.

Problem 2) (10 points)



a) The picture above shows a contour map of a function $f(x, y)$ of two variables. This function has 12 critical points and all of them are marked. Each of them is either a local max, a local min or a saddle point. The picture shows also some gradient vectors. Count the number of critical points in the following table. No justifications are necessary.

The function $f(x, y)$ has		local maxima
The function $f(x, y)$ has		local minima
The function $f(x, y)$ has		saddle points

b) (4 points) Match the following partial differential equations with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_{xx} + u_{yy} = 0$
	$u_{xx} - u_{yy} = 0$

Enter A,B,C,D here	PDE
	$u_x - u_{yy} = 0$
	$u_x - u_y = 0$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation

Problem 3) (10 points)

Find the cos of the angle between the sphere

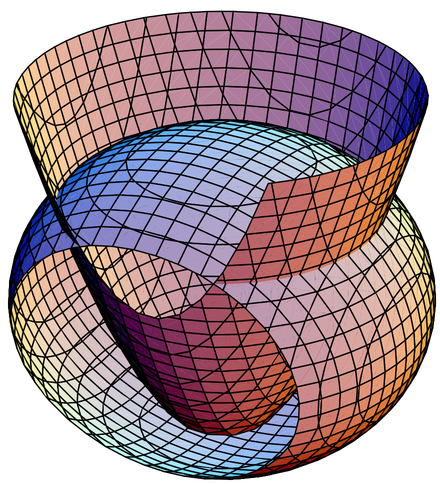
$$x^2 + y^2 + z^2 - 9 = 0$$

and the paraboloid

$$z - x^2 - y^2 + 3 = 0$$

at the point $(2, -1, 2)$.

Note: The angle between two general surfaces at a point P is defined as the angle between the tangent planes at the point P.



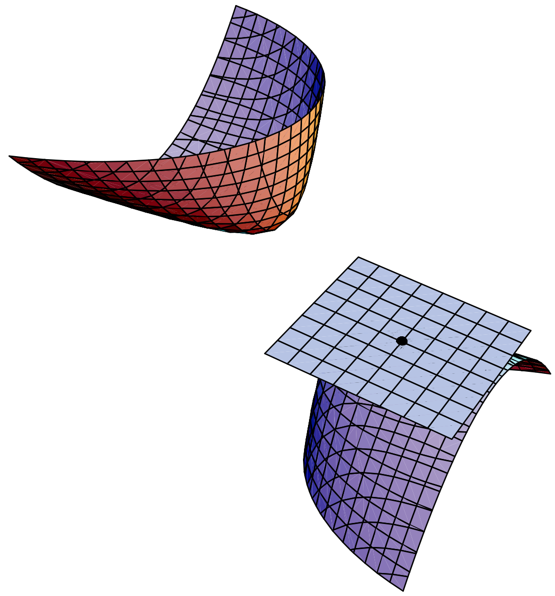
Problem 4) (10 points)

a) You know that

$$-2x + 5y + 10z = 2$$

is the equation of the tangent plane to the graph of $f(x, y)$ at the point $(-1, 2, -1)$. Find the gradient $\nabla f(-1, 2)$ at the point $(-1, 2)$ and Estimate $f(-0.998, 2.0001)$ using linear approximation.

b) Let $f(x, y, z) = x^2 + 2y^2 + 3xz + 2$. Find the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point $(2, 0, -1)$ and estimate $f(2.001, 0.01, -1.0001)$.



Problem 5) (10 points)

a) (4 points) Find all the critical points of the function $f(x, y) = xy$ in the interior of the elliptic domain

$$x^2 + \frac{1}{4}y^2 < 1 .$$

and decide for each point whether it is a maximum, a minimum or a saddle point.

b) (4 points) Find the extrema of f on the boundary

$$x^2 + \frac{1}{4}y^2 = 1 .$$

of the same domain.

c) (2 points) What is the global maximum and minimum of f on $x^2 + \frac{1}{4}y^2 \leq 1$.

Problem 6) (10 points)

a) Assume $f(x, y) = e^{2x-y-2} + y + \sin(x - 1)$ and $x(t) = \cos(5t), y(t) = \sin(5t)$. What is

$$\frac{d}{dt} f(x(t), y(t))$$

at time $t = 0$.

b) The relation

$$xyz + z^3 + xy + yz^2 = 4$$

defines z as a function of x and y near $(x,y,z) = (1,1,1)$. Find the gradient

$$\langle \frac{\partial z}{\partial x}(1,1), \frac{\partial z}{\partial y}(1,1) \rangle$$

of $z(x,y)$ at the point $(1,1)$.

Problem 7) (10 points)

The temperature in a room is given by $T(x,y,z) = x^2 + 2y^2 - 3z + 1$.

a) Barry B. Benson is hovering at the point $(1,0,0)$ and feels cold. Which direction should he go to heat up most quickly? Make sure that your answer is a unit vector.

b) At some later time, Barry arrives at the point $(3,2,1)$ and decides that this is a nice temperature. Find a direction (a unit vector) in which he can go, to stay at the same temperature and the same altitude.

Problem 8) (10 points)

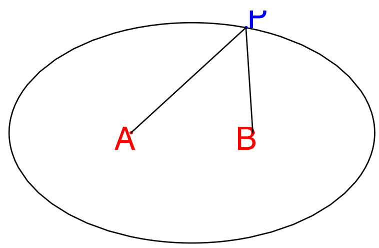
Let $g(x,y)$ denote the distance of a point $P = (x,y)$ to a point A and $h(x,y)$ the distance from P to a point B. The set of points (x,y) for which $f(x,y) = g(x,y) + h(x,y)$ is constant, forms an ellipse. In other words, the level curves of f are ellipses.

a) (4 points) Why is $\nabla g + \nabla h$ perpendicular to the ellipse?

b) (3 points) Show that if $\vec{r}(t)$ parametrizes the ellipse, then $(\nabla g + \nabla h) \cdot \vec{r}' = 0$ or $\nabla g \cdot r' = -\nabla h \cdot r'$.

c) (3 points) Conclude from this that the lines AP and BP make equal angles with the tangent to the ellipse at P. (Hint: check that $|\nabla f| = |\nabla g| = 1$).

You have now shown that light rays originating at focus A will be reflected from the ellipse to focus at the point B.

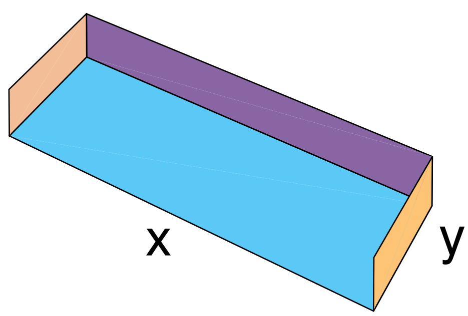


Problem 9) (10 points)

Minimize the material cost of an office tray

$$f(x, y) = xy + 2x + 2y$$

of length x, width y and height 1 under the constraint that the volume $g(x, y) = xy$ is constant and equal to 4.



Problem 10) (10 points)

A beach wind protection is manufactured as follows. There is a rectangular floor $ACBD$ of length a and width b. A pole of height c is located at the corner C and perpendicular to the ground surface. The top point P of the pole forms with the corners A and C one triangle and with the corners B and C an other triangle. The total material has a fixed area of $g(a, b, c) = ab + ac/2 + bc/2 = 12$ square meters. For which dimensions a, b, c is the volume $f(a, b, c) = abc/6$ of the tetrahedral protected by this configuration maximal?