10/4/2011 FIRST HOURLY **Math 21a, Fall 2011**

Name:

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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, and 5, we need to see **details** of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False (TF) questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | **F** | If two planes $ax + by + cz = d$ and $Ax + By + Cz = D$ are parallel then $a = A, b = B$, and $c = C$.

Solution:
We can have $A = 2a, B = 2b, C = 2c, D = 2d$ for example.

2) | **T** | F | The point $(x, y, z) = (1, 1, \sqrt{2})$ has the spherical coordinates $(\rho, \theta, \phi) = (2, \pi/4, \pi/4)$.

Solution:
Use the transformation formula.

3) | **T** | F | Every point on the parametric curve $\vec{r}(t) = \langle t, t^2, -t \rangle$ lies on the surface $xz + y = 0$.

Solution:
Check with $x = t, y = t^2, z = -t$.

4) | **T** | F | The two surfaces $f(x, y, z) = 3$ and $f(x, y, z) = 5$ of the function $f(x, y, z) = 2x^2 + y^3 + z^4$ do not intersect at any point in space.

Solution:
The function is smooth so that level surfaces to different values can not intersect.

5) | T | **F** | $\vec{u} \times \vec{i}$ and $\vec{u} \times \vec{j}$ are perpendicular for all vectors $\vec{u}$.

Solution:
Take $\vec{u} = \vec{i} + \vec{j}$.

6) | T | **F** | If $\vec{u}$ and $\vec{v}$ are parallel then $\vec{u} \cdot \vec{v} \geq |\vec{u} \times \vec{v}|$.

Solution:
We can have $\vec{u} = -\vec{v}$ in which case the left hand side is negative if $\vec{v}$ has positive length.

7) T **F** If a surface has the property that all intersections with the planes $y = constant$ are straight lines, then the surface is a plane.

Solution:
Take the function $y = x^2$ for example. Its graph is not a plane but $y = constant$ are lines.

8) T **F** For any non-zero vectors $\vec{u}$ and $\vec{w}$, we must have $\text{proj}_{\vec{u}}\vec{w} = -\text{proj}_{\vec{w}}\vec{u}$.

Solution:
The projection onto $\vec{u}$ is parallel to $\vec{u}$ and the projection onto $\vec{v}$ is parallel to $\vec{v}$.

9) T **F** In the parametric surface $\vec{r}(s,t) = \langle \sqrt{1+e^t}\cos(s)\,,\, \sqrt{1+e^t}\sin(s)\,,\, t \rangle$ the grid curves with constant s are ellipses.

Solution:
Take $s = 0$ to get the curve $\langle \sqrt{1+e^t}, 0, t \rangle$ which is the graph of a function of one variable in the xz plane.

10) T **F** There is a vector $\vec{v}$ with the property that $\vec{v} \times \langle 1,1,1 \rangle = \langle 0,0,1 \rangle$.

Solution:
Whatever the vector is, the right hand side would be perpendicular to $\langle 1,1,1 \rangle$.

11) **T** F We can assign a value $f(0,0)$ such that the function $f(x,y) = (x^3+y^3)/(x^2+y^2)$ is continuous at $(0,0)$.

Solution:
Use polar coordinates to get $f = r^3(\cos^3(\theta) + \sin^3(\theta))/r^2 = r(\cos^3(\theta) + \sin^3(\theta))/r$.

12) **T** F The curvature of a curves $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{R}(t) = \langle t^2, t^4, t^6 \rangle$ are the same at $t = 1$.

Solution:
Curvature is independent of the parametrization.

13) | T | F |

The curve given in spherical coordinates as $\phi = \pi/2, \rho = \pi/2$ is a circle.

Solution:
The result is zero

14) | T | F |

Two nonparallel planes with normal vectors $\vec{n}, \vec{m}$ intersect in a line parallel to $\vec{n} \times \vec{m}$.

Solution:
Make a picture

15) | T | F |

If $f(x, y) = x^3/3 - y^2$, then the graph of the function $f_x(x, y)$ is called a hyperbolic paraboloid.

Solution:
The partial derivative is $f(x, y) = x^2$ and $z = x^2$ is not a hyperbolic paraboloid. It is a parabolic cylinder.

16) | T | F |

The equation $\rho\cos(\theta)\sin(\phi) = 2$ in spherical coordinates defines a plane.

Solution:
In spherical coordinates, we have $x = \rho\cos(\theta)\sin(\phi)$.

17) | T | F |

The vector $\langle 3, -2\rangle$ in the two dimensional plane is perpendicular to the line $3x - 2y = 7$.

Solution:
It is the gradient $\langle 1, 2\rangle$.

18) | T | F |

The volume of the parallelepiped spanned by the vectors $\langle 1, 0, 0\rangle, \langle 0, 2, 0\rangle$ and $\langle 1, 1, 1\rangle$ is 2.

Solution:
Compute the triple scalar product which is 2.

19) | T | F | If $\vec{r}(t)$ is a curve and $|\vec{r}\,'(t)| > 0$ and $|\vec{T}'| > 0$, we have $\vec{T}(t) \cdot (\vec{N}(t) \times \vec{B}(t)) = 1$.

Solution:
The three vectors are defined and are all perpendicular to each other and have length 1. They span a cube of volume 1.

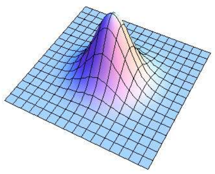
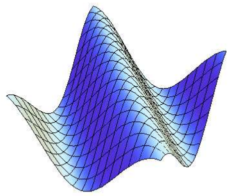
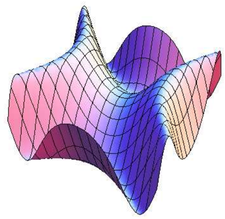
20) | T | F | The arc lengths of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{R}(t) = \langle t^2, t^4, t^6 \rangle$ are the same for $0 \leq t \leq 1$.

Solution:
This is an important property of arc length.

[ ] Total

Problem 2) (10 points)

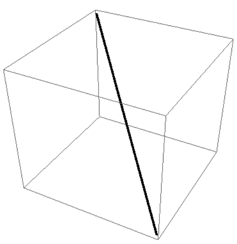
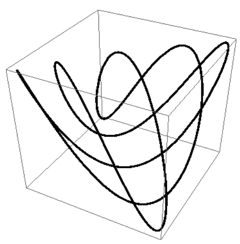
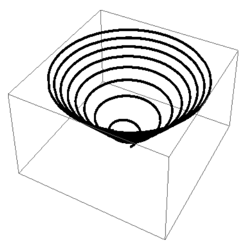
a) (2 points) Match the graphs $z = f(x,y)$ with the functions. Enter O, if there is no match. In each of the problems a) - d), each entry O,I,II,III appears exactly once.

I II III

Function $f(x,y) =$	O,I,II or III
$e^{-x^2-y^2}$	
$\cos(x+y)$	
$\sin(x^2-y^2)$	
x^4+y^4	

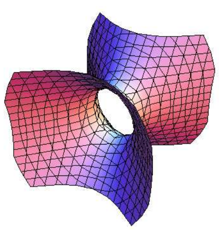
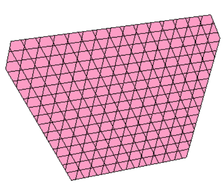
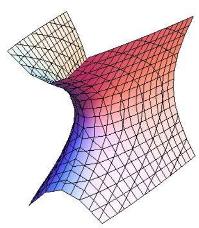
b) (3 points) Match the space curves with their parametrizations $\vec{r}(t)$. Enter O, if there is no match.

I II III

Parametrization $\vec{r}(t) =$	O, I,II,III
$\langle 1+t, 1-t, t\rangle$	
$\langle t\cos(t^2), t\sin(t^2), t\rangle$	
$\langle t, t, \sin(t^3)\rangle$	
$\langle \cos(3t), \sin(2t), \sin(5t)\rangle$	

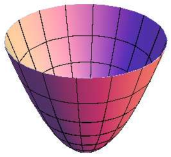
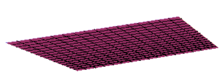
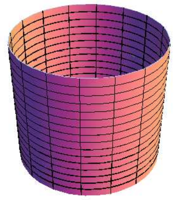
c) (2 points) Match the functions g with the level surface $g(x,y,z) = 1$. Enter O, where no match.

I II III

$g(x,y,z) =$	O, I,II,III
$(x-1)^2 - y^2 + z^2 = 1$	
$(x-1)^2 + y + z^2 = 1$	
$(x-1) + y + z = 1$	
$(x-1)^2 - y - z^2 = 1$	

d) (3 points) Match the surface with the parametrization. Enter O, where no match.

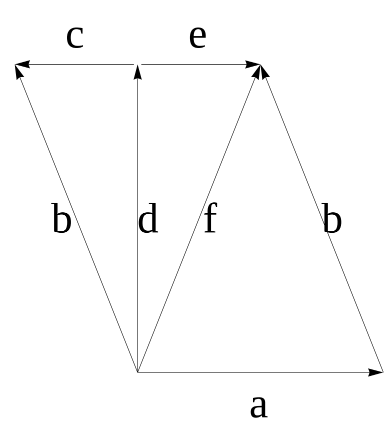
I II III

Parametriztion $\vec{r}(s,t) =$	O,I,II,III
$\langle s\cos(t), s\sin(t), s^2\rangle$	
$\langle t-1, s, s+t\rangle$	
$\langle \cos(t), \sin(t), s\rangle$	
$\langle s\cos(t), s\sin(t), s^2\sin(t)\rangle$	

Solution:
a) I,II,III,O
b) I,III,O,II
c) I,O,II,III
d) I,II,III,O

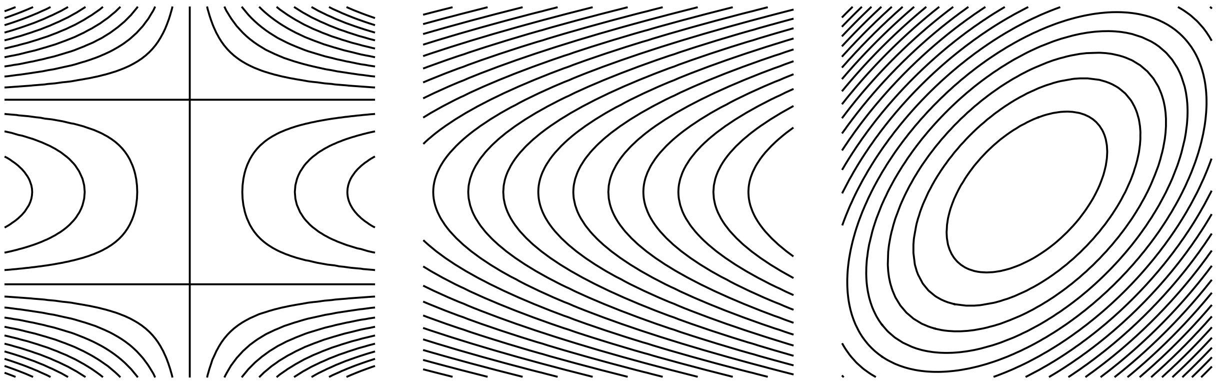
Problem 3) (10 points)

a) (7 points) Each of the vectors $a, b, c, d, e, f, 0$ (written without arrows for clarity) will appear in the blanks exactly once. As the picture indicates, you know $d \cdot e = d \cdot c = 0$.



the vector	is equal to
$\text{proj}_d f$	
$f - d$	
$-2c$	
$d - c$	
$-e$	
$\text{proj}_d e$	
$d + c$	

b) (3 points) Match the contour maps with the functions



I II III

Function $f(x,y) =$	Enter O,I,II or III
$y - x$	
$(y^2 - 1)x$	
$y^2 + x^2 - xy$	
$y^2 - x$	

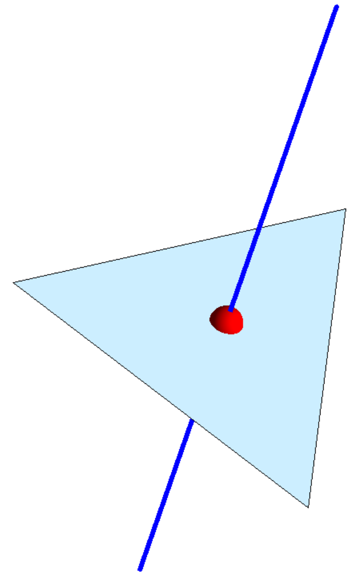
Solution:
Its obviously a "deaf cob":

the vector	is equal to
$\text{proj}_{\vec{d}}\vec{f}$	$\vec{d}$
$\vec{f} - \vec{d}$	$\vec{e}$
$-2\vec{c}$	$\vec{a}$
$\vec{d} - \vec{c}$	$\vec{f}$
$-\vec{e}$	$\vec{c}$
$\text{proj}_{\vec{d}}\vec{e}$	$\vec{o}$
$\vec{d} + \vec{c}$	$\vec{b}$

b) O,I,III,II

Problem 4) (10 points)

a) (4 points) The **center** of the triangle $A = (3,2,1), B = (1,1,1), C = (2,0,4)$ is the point $P = (A+B+C)/3 = (2,1,2)$. Find the line L perpendicular to the plane which contains A, B, C and which goes through P.



b) (3 points) Find the equation of the plane through A, B, C.

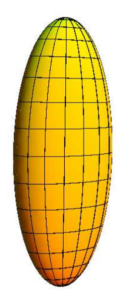
c) (3 points) Find the area of the triangle ABC.

Solution:
The vectors $\vec{BA} = \langle 2,1,0\rangle, \vec{BC} = \langle 1,-1,3\rangle$ are in the plane. Their cross product $\vec{n} = \langle 3,-6,-3\rangle$ gives the direction normal to the plane as well as the direction of the line.
a) The equation of the line is $\vec{OP} + t\vec{n} = \langle 2,1,2\rangle + t\langle 3,-6,3\rangle$.
b) The equation of the plane is $3x - 6y - 3z = 0$.
c) The area of the triangle is half the area of the parallelogram which is the length of the cross product divided by 2. This is $3\sqrt{6}/2$.

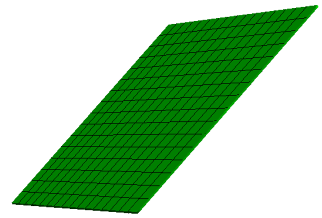
Problem 5) (10 points)

Complete the parametrizations:

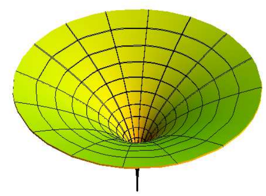
a) (3 points) $\vec{r}(u,v) = \langle 2 + 3\cos(u)\sin(v), 3 + \sin(u)\sin(v),$ [ ] $\rangle$ parametrizes the ellipsoid $(x-2)^2/9 + (y-3)^2 + (z-5)^2/16 = 1$.



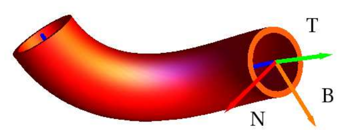
b) (2 points) $\vec{r}(u,v) = \langle u, v, \boxed{} \rangle$ parametrizes the plane $x + y + z = 1$.



c) (3 points) $\vec{r}(u,v) = \langle v^3\cos(u), \boxed{}, v\rangle$ parametrizes the surface of revolution $x^2 + y^2 = z^6$.



d) (2 points) $\vec{r}(u,v) = \vec{r}(v) + \cos(u)\vec{N}(v) + \sin(u)\boxed{}$ parametrizes a tube around a curve $\vec{r}(v)$ which has unit tangent vector $\vec{T}(v)$, normal vector $\vec{N}(v)$ and binormal vector $\vec{B}(v)$.



Solution:

a) This is an ellipsoid centered at $(2,3,5)$ which is deformed. The last entry is $\boxed{5 + 4\cos(v)}$.

b) This is a plane. Solve for $z = 1 - x - y$ and use u, v to get $\boxed{1 - u - v}$.

c) This is a surface of revolution and we have $\boxed{v^3\sin(u)}$.

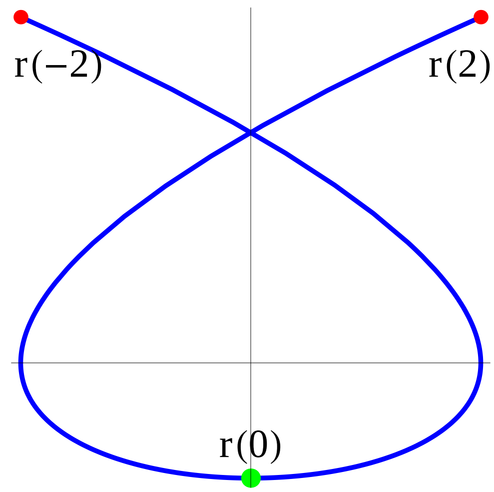
d) Since the tube circular and the grid curves with constant v are circles, we can use N, B to draw the circle. This is almost completed. W only have to fill in $\boxed{\vec{B}(v)}$. This example is actually one of the main motivations to use the TNB frame. It allows to draw beautiful tubes around a given curve.

Problem 6) (10 points)

We look at the parametrized curve

$$\vec{r}(t) = \langle \frac{t^3}{3} - t, t^2 - 1, 0\rangle$$

whose image you see in the picture showing it in the xy plane for $-2 \leq t \leq 2$.



a) (3 points) Find the velocity $\vec{r}\,'(t)$, the acceleration $\vec{r}\,''(t)$ and speed $|\vec{r}\,'(t)|$.

b) (2 points) Evaluate this at $t = 0$ to get $\vec{r}\,'(0), \vec{r}\,''(0)$ and $|\vec{r}\,'(0)|$.

c) (2 points) Find the curvature $|\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$ at $(0, -1, 0)$.

d) (3 points) Find the arc length of the curve $\vec{r}(t)$ from $-2 \leq t \leq 2$.

Solution:
a) $\vec{r}\ '(t) = \langle t^2 - 1, 2t, 0\rangle$. $\vec{r}\ ''(t) = \langle 2t, 2, 0\rangle$ and $|\langle t^2 - 1, 2t, 0\rangle| = \sqrt{t^4 + 2t + 1} = \sqrt{(t^2 + 1)^2} = t^2 + 1$.
b) $\vec{r}\,'(0) = \langle -1, 0, 0\rangle, \vec{r}\,''(0) = \langle 0, 2, 0\rangle$ and $|\vec{r}\,'(0) = 1$.
c) Since the speed is 1, the curvature is

$$|\langle -1, 0, 0\rangle \times \langle 0, 2, 0\rangle| = |\langle 0, 0, -2\rangle| = 2$$

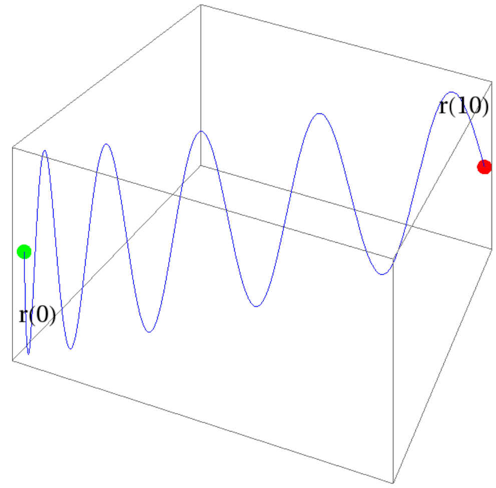
d) The arc length is $\int_{-2}^{2} |\vec{r}\,'(t)|\; dt = \int_{-2}^{2} t^2 + 1\; dt = t^3/3 + t|_{-2}^{2} = 28/3$.

Problem 7) (10 points)

a) (4 points) We know $\vec{r}\,''(t) = \langle 1, 2, \pi^2 \sin(\pi t)\rangle$ and the initial velocity $\vec{r}\,'(0) = \langle 1, 0, -\pi\rangle$. Find $\vec{r}\,'(t)$.

b) (3 points) Assume we know also $\vec{r}(0) = \langle 0, 0, 10\rangle$. Find $\vec{r}(10)$.

c) (3 points) What is the projection of $\vec{r}\;'(10)$ onto $\langle 1, 1, 0\rangle$?



Solution:
a) Integrate to get

$$\vec{r}\,'(t) = \langle t, 2t, -\pi\cos(\pi t)\rangle + \langle c_1, c_2, c_3\rangle \; .$$

Comparing the initial velocity gives the constants and so

$$\vec{r}'(t) = \langle 1 + t, 2t, -(\pi\cos(\pi t))\rangle \; .$$

b) Integrate again and compare coefficients to get

$$\vec{r}(t) = t + t^2/2, t^2, 10 - \sin(\pi t) \; .$$

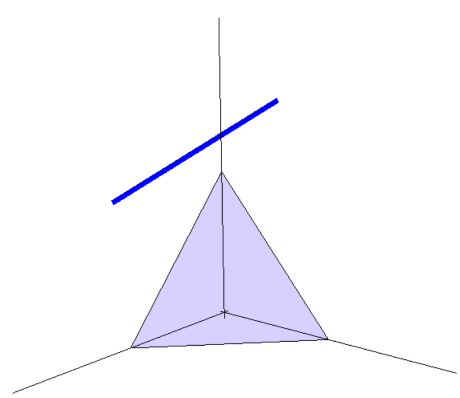
we have $r(10) = \langle 60, 100, 10\rangle$.
c) We have $\vec{r}\;'(10) = \langle 10, 20, -\pi\rangle$. The vector projection is $\vec{r}\;'(10) \cdot \langle 1, 1, 0\rangle = (31/2)\langle 1, 1, 0\rangle$

Problem 8) (10 points)

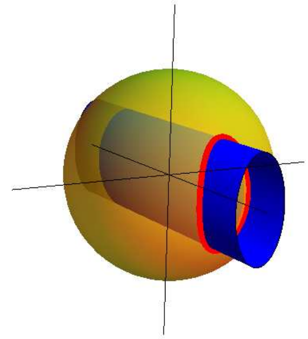
a) (5 points) Find the distance between the plane $x + y + z = 1$ and the line

$$x - 1 = \frac{(y-1)}{-2} = z - 1$$

which is parallel to the plane.
(You do not have to check that it is parallel).



b) (5 points) The intersection of the cylinder $4x^2+z^2 = 1$ with the sphere centered at $(0,0,0)$ with radius $\rho = \sqrt{2}$ cuts out two curves. Parametrize the curve which contains the point $(0,1,1)$.



Solution:
a) Choose a point on the plane $P = (1,0,0)$ and a point $Q = (1,1,1)$ on the line and compute the distance from P to the plane We have $\vec{PQ} = \langle 0,1,1 \rangle$ and

$$d = \frac{|\langle 0,1,1 \rangle \cdot \langle 1,1,1 \rangle|}{|\langle 1,1,1 \rangle|} = 2/\sqrt{3} \, .$$

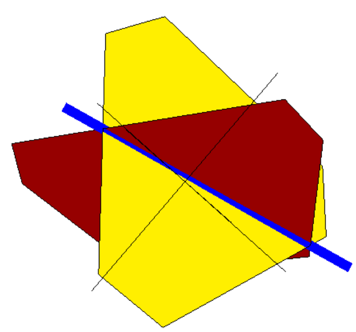
b) First parametrize the ellipse in the xz-plane with $\vec{r}(t) = \langle \cos(t)/2, ..., \sin(t) \rangle$ where we do not know y yet. The sphere has the equation $x^2 + y^2 + z^2 = 2$ and we can solve for y and get the parametrization

$$\vec{r}(t) = \langle \cos(t)/2, \sqrt{2 - \cos^2(t)/4 - \sin^2(t)}, \sin(t) \rangle \, .$$

Problem 9) (10 points)

a) (5 points) Find a parametrization of the intersection line L of the two planes

$$\begin{aligned} 2x - 2y + z &= 1 \, , \\ x + y + z &= 1 \, . \end{aligned}$$



b) (5 points) Find the symmetric equation for the line M parallel to the line L computed in a) which passes through $(1,2,3)$.

Solution:
a) A point in the intersection of the plane is $P = (0, 0, 1)$. The cross product between the normal vectors is $\langle -3, -1, 4\rangle$. The parametrization of the line is $\vec{r}(t) = \langle 0, 0, 1\rangle + t\langle -3, -1, 4\rangle$.
b) Now translate the line through $(1, 2, 3$ to get $\vec{r}(t) = \langle 1, 2, 3\rangle + t\langle -3, -1, 4\rangle$ which has the symmetric equations

$$\frac{x-1}{(-3)} = \frac{y-2}{(-1)} = \frac{x-3}{4} .$$