

Name:

|                             |
|-----------------------------|
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| MWF 10 Andrew Cotton-Clay   |
| MWF 10 Oliver Knill         |
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| MWF 11 Ana Caraiani         |
| MWF 11 Chris Phillips       |
| MWF 11 Ethan Street         |
| MWF 12 Toby Gee             |
| MWF 12 Xinwen Zhu           |
| TTH 10 Jack Huizenga        |
| TTH 10 Fred van der Wyck    |
| TTH 11:30 Ming-Tao Chuan    |
| TTH 11:30 Fred van der Wyck |

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or un-staple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

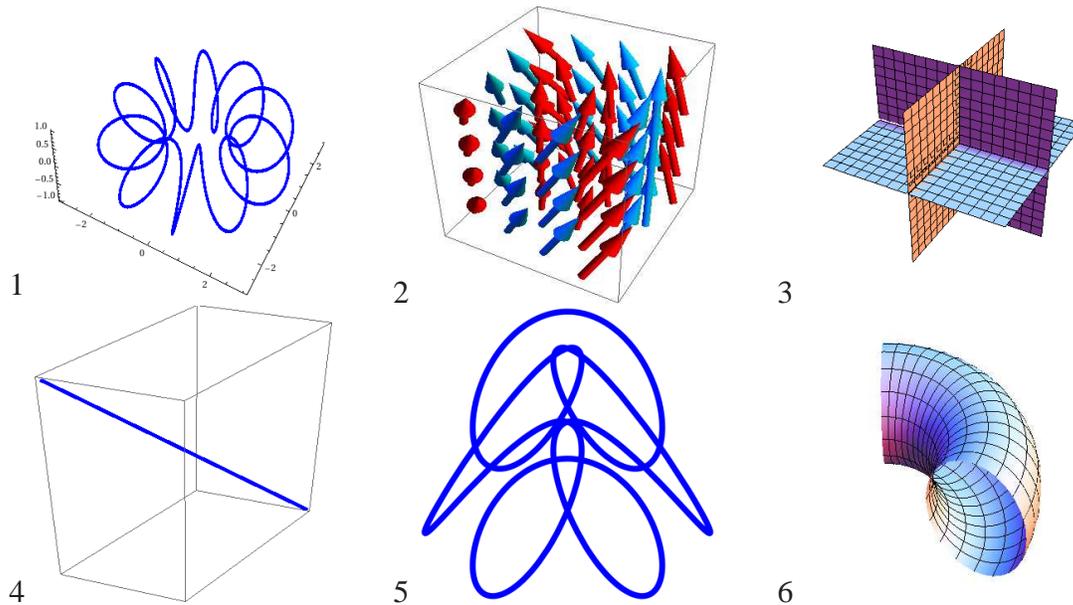
|        |  |     |
|--------|--|-----|
| 1      |  | 20  |
| 2      |  | 10  |
| 3      |  | 10  |
| 4      |  | 10  |
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| 7      |  | 10  |
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| 10     |  | 10  |
| 11     |  | 10  |
| 12     |  | 10  |
| 13     |  | 10  |
| Total: |  | 140 |

Problem 1) True/False questions (20 points). No justifications are needed.

- 1)  T  F The functions  $e^{x^2+y^3-y}$  and  $x^2 + y^3 - y$  have the same critical points.
- 2)  T  F The line  $\vec{r}(t) = \langle t^2, t^2, t^2 \rangle$  hits the plane  $x + y + z = 100$  at a right angle.
- 3)  T  F The quadric  $x^2 - 2y^2 + z^2 = 5$  is a one sheeted hyperboloid.
- 4)  T  F The relation  $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$  is only possible if at least one of the vectors  $\vec{u}$  and  $\vec{v}$  is the zero vector.
- 5)  T  F The partial differential equation  $u_t + u_x u = u_{tt}$  is called the **Burgers equation**.
- 6)  T  F The curvature of the curve  $\vec{r}(t) = \langle \sin(2t), \cos(2t)/\sqrt{2}, \cos(2t)/\sqrt{2} \rangle$  at  $t = 0$  is equal to the curvature of the curve  $\vec{s}(t) = \langle 0, \cos(3t), \sin(3t) \rangle$  at  $t = 0$ .
- 7)  T  F The space curve  $\vec{r}(t) = \langle \sin(t), t^2, \cos(t) \rangle$  for  $t \in [0, 10\pi]$  is located on a cylinder.
- 8)  T  F If a smooth function  $f(x, y)$  has a global maximum, then it has a global minimum.
- 9)  T  F If  $L(x, y)$  is the linearization of  $f(x, y)$  at  $(x_0, y_0)$  and  $\vec{s}(t)$  is the line tangent to the curve  $\vec{r}(t)$  at the point  $\vec{r}(t_0) = \vec{s}(t_0) = (x_0, y_0)$  so that  $|\vec{r}'(t_0)| = |\vec{s}'(t_0)| = 1$ , then  $|d/dt L(\vec{s}(t))| = |d/dt f(\vec{r}(t))|$  at the time  $t = t_0$ .
- 10)  T  F If  $\vec{F}$  is a gradient field and  $\vec{r}(t)$  is a flow line defined by  $\vec{r}'(t) = \vec{F}(\vec{r}(t))$ , then the line integral  $\int_0^1 \vec{F} \cdot d\vec{r}$  is either positive or zero.
- 11)  T  F The flux of the vector field  $\vec{F} = \nabla f$  through the surface  $f(x, y, z) = x^4 + y^4 + z^4 = 1$  is positive if the surface is oriented so that  $\vec{r}_u \times \vec{r}_v$  points in the direction of the gradient of  $f$ .
- 12)  T  F If we extremize the function  $f(x, y)$  under the constraint  $g(x, y) = 1$ , and the functions are the same  $f = g$ , all points on the constraint curve are extrema for  $f$ .
- 13)  T  F If a point  $(x_0, y_0)$  is a minimum of  $f(x, y)$  under the constraint  $g(x, y) = 1$ , then it is also a local minimum of the function  $f(x, y)$  without constraints.
- 14)  T  F If a vector field  $\vec{F}(x, y)$  is a gradient field, then any line integral along any closed ellipse is zero.
- 15)  T  F The flux of an irrotational vector field is zero through any surface  $S$  in space.
- 16)  T  F The divergence of a gradient field  $\vec{F}(x, y, z) = \nabla f(x, y, z)$  is everywhere zero.
- 17)  T  F The line integral of the vector field  $\vec{F}(x, y, z) = \langle x, y, z \rangle$  along a circle in the  $xy$ -plane is zero.
- 18)  T  F For any solid  $E$ , the moment of inertia  $\iiint_E x^2 + y^2 dx dy dz$  is always larger than the volume  $\iiint_E 1 dx dy dz$  of  $E$ .
- 19)  T  F The curvature of a parametrized curve satisfying  $|\vec{r}'(t)| = 1$  is bounded above by the length  $|\vec{r}''(t)|$  of the acceleration.
- 20)  T  F Given a vector field  $\vec{F} = \langle P, Q, R \rangle$ , the directional derivative of  $\text{div}(\vec{F}(x, y, z))$  in the direction  $\vec{v} = \langle 1, 0, 0 \rangle$  is  $P_{xx} + Q_{xy} + R_{xz}$ .

Problem 2) (6 points)

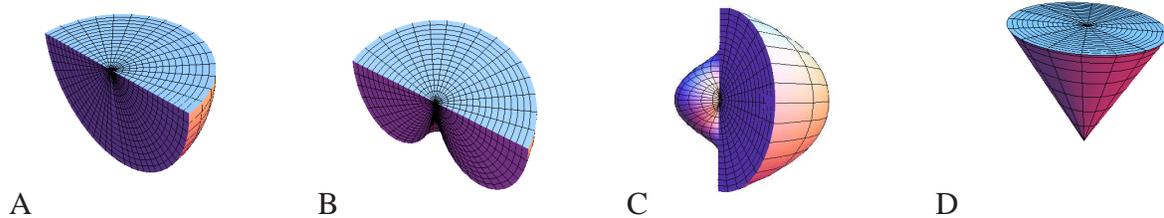
a) (6 points) Match the objects with their definitions



| Enter 1-6 | Object definition                                                                          |
|-----------|--------------------------------------------------------------------------------------------|
|           | $\vec{r}(t) = \langle (2 + \cos(10t)) \cos(t), (2 + \cos(10t)) \sin(t), \sin(10t) \rangle$ |
|           | $\vec{F}(x, y, z) = \langle -y, x, 2 \rangle$                                              |
|           | $\vec{r}(t, s) = \langle (2 + \cos(s)) \cos(t), (2 + \cos(s)) \sin(t), \sin(s) \rangle$    |
|           | $x^2 y^2 z^2 = 0$                                                                          |
|           | $(x - 1)/5 = (y - 2)/10 = (z - 1)/3$                                                       |
|           | $\vec{r}(t) = \langle \sin(t) + \cos(5t), \cos(t) + \cos(6t) \rangle$                      |

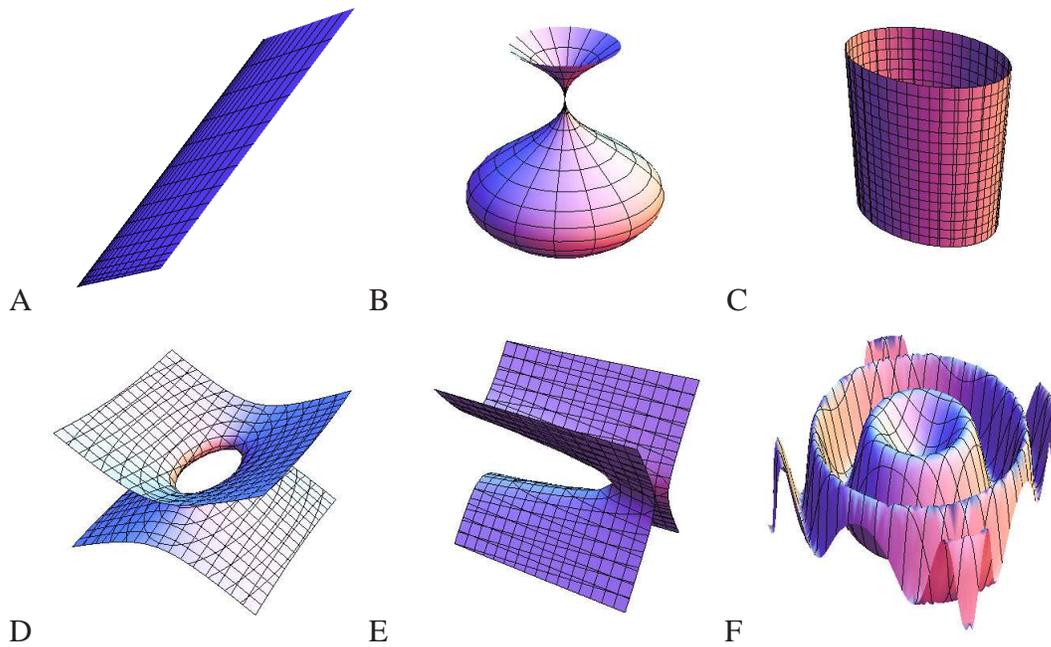
b) (4 points) Match the solids with the triple integrals:

| Enter A-D | 3D integral computing volume                                                                   |
|-----------|------------------------------------------------------------------------------------------------|
|           | $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{1/\cos(\phi)} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$  |
|           | $\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^{\sin(\phi)} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$ |
|           | $\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^1 \rho^2 \sin(\phi) \, d\rho d\phi d\theta$            |
|           | $\int_0^{2\pi} \int_0^{\pi} \int_0^{\theta} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$          |



Problem 3) (10 points)

a) (6 points) The surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.

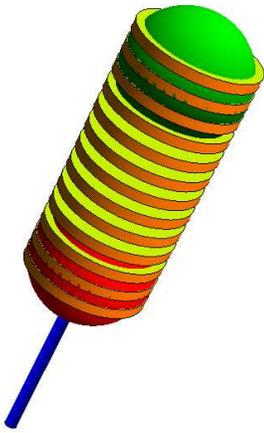


| Enter A-F here | Function or parametrization                                                       |
|----------------|-----------------------------------------------------------------------------------|
|                | $\vec{r}(u, v) = \langle u^2, v^2, u^2 + v^2 \rangle$                             |
|                | $\vec{r}(u, v) = \langle (1 + \sin(u)) \cos(v), (1 + \sin(u)) \sin(v), u \rangle$ |
|                | $4x^2 + y^2 - 9z^2 = 1$                                                           |
|                | $x - 9y^2 + 4z^2 = 1$                                                             |
|                | $\vec{r}(u, v) = \langle u, v, \sin(u^2 + v^2) \rangle$                           |
|                | $4x^2 + 9y^2 = 1$                                                                 |

b) (4 points) If the blank box is replaced by  $\nabla f(5, 6)$  the statement becomes true or false. Determine which case we have. The function  $f(x, y)$  is an arbitrary nice function like for example  $f(x, y) = x - yx + y^2$ . The curve  $\vec{r}(t)$ , wherever it appears, parametrizes the level curve  $f(x, y) = f(5, 6)$  and has the property that  $\vec{r}'(0) = \langle 5, 6 \rangle$ .

| True/False | Topic                  | Statement                                                                                                    |
|------------|------------------------|--------------------------------------------------------------------------------------------------------------|
|            | Linearization          | $L(x, y) = f(5, 6) + \boxed{\phantom{000}} \cdot \langle x - 5, y - 6 \rangle$                               |
|            | Chain rule             | $\frac{d}{dt} f(\vec{r}(t)) _{t=0} = \boxed{\phantom{000}} \cdot \vec{r}'(0)$                                |
|            | Steepest descent       | $f$ decreases at $(5, 6)$ most in the direction of $\boxed{\phantom{000}}$                                   |
|            | Estimation             | $f(5 + 0.1, 5.99) \sim f(5, 6) + \boxed{\phantom{000}} \cdot \langle 0.1, -0.01 \rangle$                     |
|            | Directional derivative | $D_{\vec{v}} f(5, 6) = \boxed{\phantom{000}} \cdot \vec{v},  \vec{v}  = 1$                                   |
|            | Level curve            | of $f$ through $(5, 6)$ has the form $\boxed{\phantom{000}} \cdot \langle x - 5, y - 6 \rangle = 0$          |
|            | Vector projection      | of $\nabla f(5, 6)$ onto $\vec{v}$ is $\vec{v}(\vec{v} \cdot \boxed{\phantom{000}}) /  \vec{v} ^2$           |
|            | Tangent line           | of $\vec{r}(t)$ at $(5, 6)$ is parametrized by $\vec{R}(s) = \langle 5, 6 \rangle + s \boxed{\phantom{000}}$ |

Problem 4) (10 points)



Two ice cream scoops given by spheres

$$x^2 + y^2 + (z + 1)^2 = 1$$

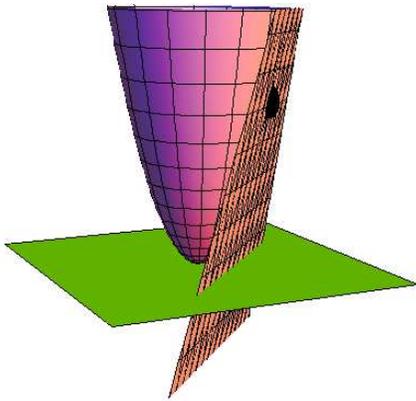
and

$$(x - 1)^2 + (y - 1)^2 + (z - 2)^2 = 1$$

are enclosed by a cylinder which is tangent to both spheres. Find the equation of the cylinder.

**Hint:** consider the distance of a general point  $(x, y, z)$  to the line passing through the centers of the spheres.

Problem 5) (10 points)



Find a parametrization

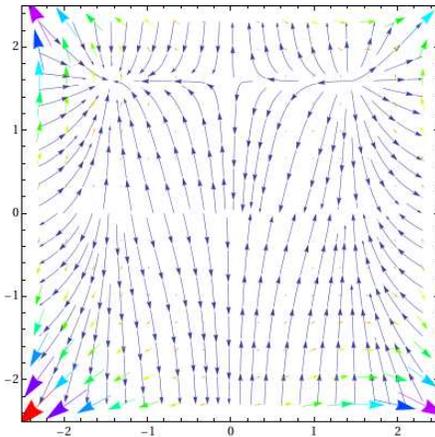
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

for the line obtained by intersecting the tangent plane  $\Sigma$  to the surface

$$x^2 + y^2 - z = 0$$

at  $(-1, -1, 2)$  with the  $xy$ -plane.

Problem 6) (10 points)



The vector field

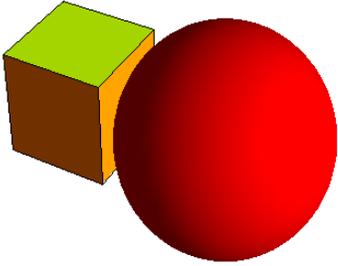
$$\vec{F}(x, y) = \langle P, Q \rangle = \langle y(x^4 - 2x^2), x(y^4 - 4y) \rangle$$

has the curl

$$f(x, y) = \text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y) .$$

Find and classify all critical points of  $f$  by deciding whether they are local maxima, local minima or saddle points. Is there a global maximum or global minimum of  $f$ ?

Problem 7) (10 points)



We want to minimize the volume of the union of a **sphere** of radius  $x$  and a **cube** of side length  $y$  under the constraint that the sum of the two surface areas is equal to 4. Find the minimal value using the Lagrange method.

Remark: You do not have to show any derivations of the volume and surface area of the sphere.

Problem 8) (10 points)

A solid  $E$  in space is determined by the inequalities

$$0 \leq z \leq 9 ,$$

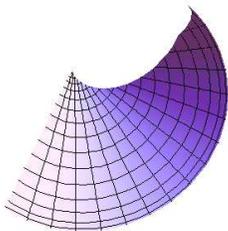
$$z^2 - x^2 - y^2 \geq 4$$

and

$$x^2 + y^2 \leq 1 .$$

Find the volume of  $E$ .

Problem 9) (10 points)



A surface  $S$  is parametrized by

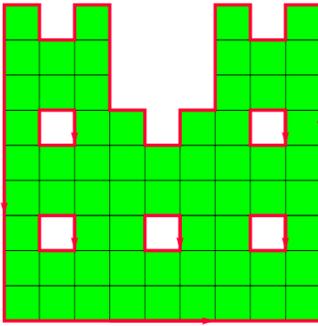
$$\vec{r}(u, v) = e^{-u^2} \langle 1, \sin(v), \cos(v) \rangle$$

where

$$0 \leq u \leq \sqrt{\pi}, u^2 \leq v \leq \pi .$$

Find its surface area.

Problem 10) (10 points)

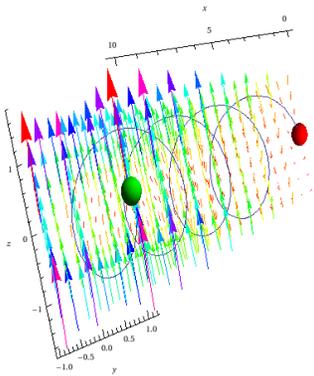


What is the line integral  $\int_C \vec{F} \cdot d\vec{r}$  of the vector field

$$\vec{F}(x, y) = \langle 1 + y + 2xy, y^2 + x^2 \rangle$$

along the boundary  $C$  of the planar “castle region” shown in the picture? Each of the 5 windows is a unit square and the base of the castle has length 9. The boundary consists of 6 curves which are all oriented so that the region is to the left.

Problem 11) (10 points)



Compute the line integral of the vector field

$$\vec{F}(x, y, z) = \langle \cos(x), 2 + \cos(y), e^z + x(y^2 + z^2) \rangle$$

along the curve  $\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$  with  $0 \leq t \leq 3\pi$ .

**Hint:** you might want to find a split  $\vec{F} = \vec{G} + \vec{H}$  and compute line integrals of  $\vec{G}$  and  $\vec{H}$  separately.

Problem 12) (10 points)



A biker in the Harvard Hemenway gym pedals. Assume that the force of a foot is

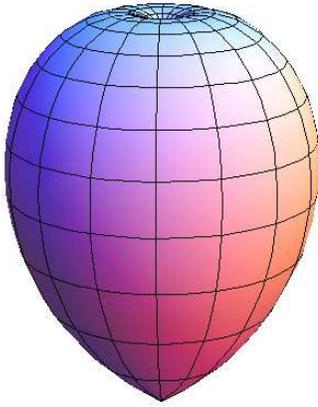
$$\vec{F} = \langle 0, 0, x^3 - x^2 + \sqrt{2 + \sin(z)} \rangle$$

and that one of the feet moves on a path  $C : \vec{r}(t) = \langle 2 \cos(t), 0, 2 \sin(t) \rangle$ . How much work

$$\int_C \vec{F} \cdot d\vec{r}$$

is done by this foot, when pedaling 10 times which means  $0 \leq t \leq 20\pi$ ?

Problem 13) (10 points)



X-Rays have intensity and direction and are given by a vector field

$$\vec{F}(x, y, z) = \langle z^7, \sin(z) + y + z^{77}, z + \cos(xy) + \sin(y) \rangle .$$

A **tonsil** is given in spherical coordinates as  $\rho \leq \phi$ . Find the flux of the X-Ray field  $\vec{F}$  through the surface  $\rho = \phi$  of the tonsil. The surface is oriented with normal vectors pointing outside. **Remark:** The flux is the amount of **ionizing radiation** absorbed by the tissue. This X-ray exposure is measured in the unit **Gray** which corresponds to the radiation amount to deposit 1 **joule** of energy in 1 **kilogram** of matter and corresponds to about 100 **Rem**. A typical dental X-ray is reported to lead to about one tenth to one half of a Rem.