Name:

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| TTH 11:30 Ming-Tao Chuan |
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- Start by printing your name in the above box and check your section in the box to the left.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

- The hourly exam itself will have space for work on each page. This space is excluded here in order to save printing resources.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [T] [F]   The directional derivative $D_u f$ is a vector normal to a level surface of f.

2) [T] [F]   At a critical point of a function f, the gradient vector has length 1.

3) [T] [F]   At a critical point (x, y) of a function f, the tangent plane to the graph of f does not exist.

4) [T] [F]   For any point (x, y) which is not a critical point, there is a unit vector $\vec{u}$ for which $D_{\vec{u}} f(x, y)$ is nonzero.

5) [T] [F]   If $f_{xx}(0, 0) = 0$, $D = f_{xx} f_{yy} - f_{xy}^2 \neq 0$, and $\nabla f(0, 0) = \langle 0, 0 \rangle$, then $(0, 0)$ is a saddle point.

6) [T] [F]   A continuous function defined on the closed unit disc $x^2 + y^2 \leq 1$ has an absolute maximum inside the disc or on the boundary.

7) [T] [F]   The function $f(x, y) = x^2 - y^2$ has a neither a local maximum nor a local minimum at $(0, 0)$.

8) [T] [F]   If (x, y) is a maximum of $f(x, y)$ under the constraint $g(x, y) = 5$ then it is also a maximum of $f(x, y) + g(x, y)$ under the constraint $g(x, y) = 5$.

9) [T] [F]   The functions $f(x, y)$ and $g(x, y) = (f(x, y))^6$ always have the same critical points.

10) [T] [F]   For $f(x, y, z) = x^2 + y^2 + 2z^2$, the vector $\nabla f(1, 1, 1)$ is perpendicular to the surface $f(x, y, z) = 4$ at the point $(1, 1, 1)$.

11) [T] [F] $f(x, y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition.

12) [T] [F]   If (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f.

13) [T] [F]   If $(1, 1, 1)$ is a maximum of f under the constraints $g(x, y, z) = c$, $h(x, y, z) = d$, and the Lagrange multipliers satisfy $\lambda = 0$, $\mu = 0$, then $(1, 1, 1)$ is a critical point of f.

14) [T] [F]   Suppose f has a maximum value at a point P relative to the constraint $g = 0$. If the Lagrange multiplier $\lambda = 0$, then P is also a critical point for f without the constraint.

15) [T] [F]   At a saddle point, all directional derivatives are zero.

16) [T] [F]   The minimum of $f(x, y)$ under the constraint $g(x, y) = 0$ is always the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

17) [T] [F]   At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \leq 0$.

18) [T] [F]   It is possible that $f(x, y)$ attains a maximum under the constraint $g(x, y) = 0$ at a point, where $\nabla f \neq \lambda \nabla g$.

19) [T] [F]   Any Lagrange problem which asks for an extremum of $f(x, y)$ under a constraint $g(x, y) = 0$ has either a maximum or a minimum.

20) [T] [F]   The function $u(x, y) = \sin(x + y)$ satisfies the PDE $u_{xx} + u_{yy} - 2u_{xy} = 0$.

Problem 2) (10 points) No justifications needed.

a) (4 points) Fill in the boxes. You do not need to give additional explanations.

Chain rule:	$\frac{d}{dt}f(\vec{r}(t)) = \boxed{} \qquad \cdot \vec{r}\,'(t)$
Directional derivative D_v	$D_{\langle 2,3\rangle/\sqrt{13}}f(1,1) = \nabla f(1,1) \cdot \boxed{}$
Linearization of $f(x,y)$ at $(1,1)$	$L(x,y) = \boxed{} \qquad + \nabla f(1,1)\cdot(x-1,y-1)$
Equation of tangent line at $(1,1)$	$\nabla f(1,1)\cdot\langle x-1, y-1\rangle = \boxed{}$
Critical point $(1,1)$ of f	$\nabla f(1,1) = \boxed{}$
Lagrange equations	$\nabla f(x,y) = \boxed{} \qquad \nabla g(x,y), \; g(x,y) = c.$
Type I integral	$\int_a^b \int_{c(x)}^{d(x)} f(x,y) \boxed{} \qquad .$
Type II integral	$\int_c^d \int_{a(y)}^{b(y)} f(x,y) \boxed{} \qquad .$
Integration in polar coordinates	$\int_a^b \int_{f(\theta)}^{g(\theta)} \boxed{} \qquad f(r\cos(\theta), r\sin(\theta))\,drd\theta.$
Area	$\int\int_R \boxed{} \qquad dxdy$
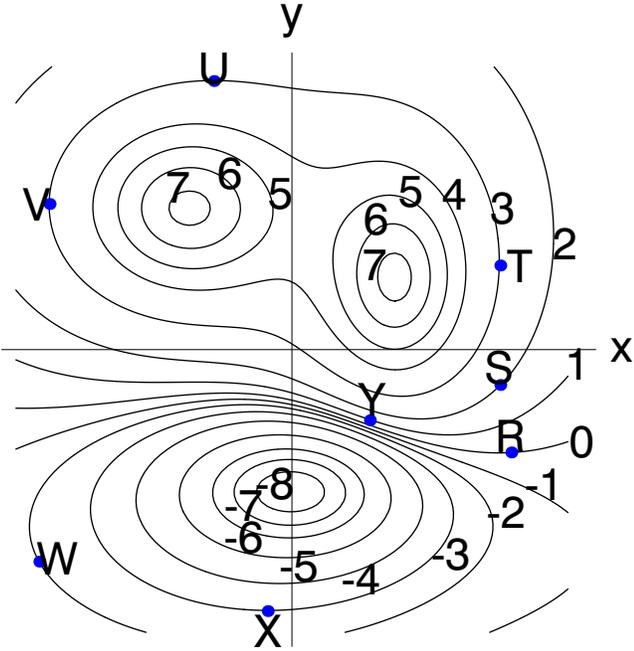
b) (2 points) Circle the point at which the magnitude of the gradient vector ∇f is greatest. Mark exactly one point. Justify your answer.

R	S	T	U	V	W	X	Y

c) (2 points) Circle the points at which the partial derivative f_x is strictly positive. Mark any number of points on this question. Justify your answers.

R	S	T	U	V	W	X	Y

d) (2 points) We know that the directional derivative in the direction $(1,1)/\sqrt{2}$ is zero at one of the following points. Which one? Mark exactly one point on this question.

R	S	T	U	V	W	X	Y

3

Problem 3) (10 points)

a) Locate and classify all the critical points of

$$f(x, y) = 3y - y^3 - 3x^2y .$$

b) Where on the parameterized surface

$$\vec{r}(x, y) = \langle u, v, w \rangle = \langle xy^3, x^2/2, 3y^2/2 \rangle$$

is the function $g(u, v, w) = u - v - w$ extremal? To investigate this, find all the critical points of the function $f(x, y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$. For each critical point, specify whether it is a local maximum, a local minimum or a saddle point and show how you know.

Problem 4) (10 points)

Evaluate the double integral

$$\int_0^4 \int_0^{y^2} \frac{x^4}{4 - \sqrt{x}} \, dx \, dy .$$
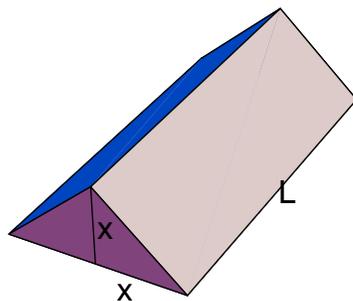
Problem 5) (10 points)

a) (6 points) Find all critical points of $f(x, y) = 3xe^y - e^{3y} - x^3$ and classify them.

b) (4 points) Does the function have a absolute maximum or absolute minimum? Make sure to justify also this answer.

Problem 6) (10 points)

We minimize the surface of a roof of height x and width $2x$ and length $L = \sqrt{2}y$ if the volume $V(x, y) = x^2\sqrt{2}y$ of the roof is fixed and equal to $\sqrt{2}$. In other words, you have to minimize $f(x, y) = 2x^2 + 4xy$ under the constraint $g(x, y) = x^2y = 1$. Solve the problem with the Lagrange method.
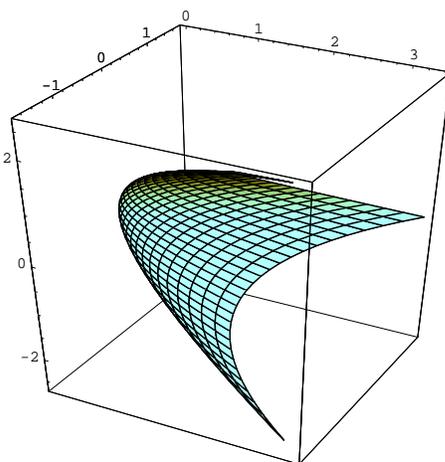
L

X

X

Problem 7) (10 points)

Find all the critical points of $f(x, y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

Problem 8) (10 points)

The temperature distribution in a room is $T(x, y, z) = x + y + z$. On which point of the parametrized surface

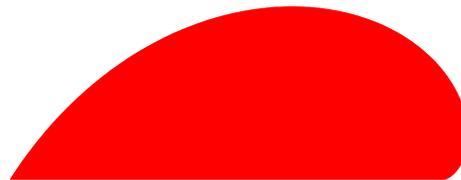$$\vec{r}(s, t) = \langle x, y, z \rangle = \langle s^2 + t^2, st, 2s - t \rangle$$

is the temperature extremal? Is it a maximum or a minimum?

Problem 9) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta^2$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}(\pi - (x^2 + y^2)^{1/4})} \, dx \, dy \ .$$

Problem 10) (10 points)

Suppose $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 1, 2)$.

a) Find the linear approximation of $f(1.01, 0.98)$.
b) What is the gradient ∇f at $(1, 1)$?
c) What is the equation $ax + by = d$ of the tangent line at $(1, 1)$?