Math21a Review

Oliver Knill, November 16, 2008

Menu

- partial differential equations
- gradient
- tangent lines / tangent planes
- linear estimation
- extrema without constraints
- implicit differentiation
- integration in 2D
- directional derivatives
- chain rule
- extrema with constraints
- Integration in polar coord.

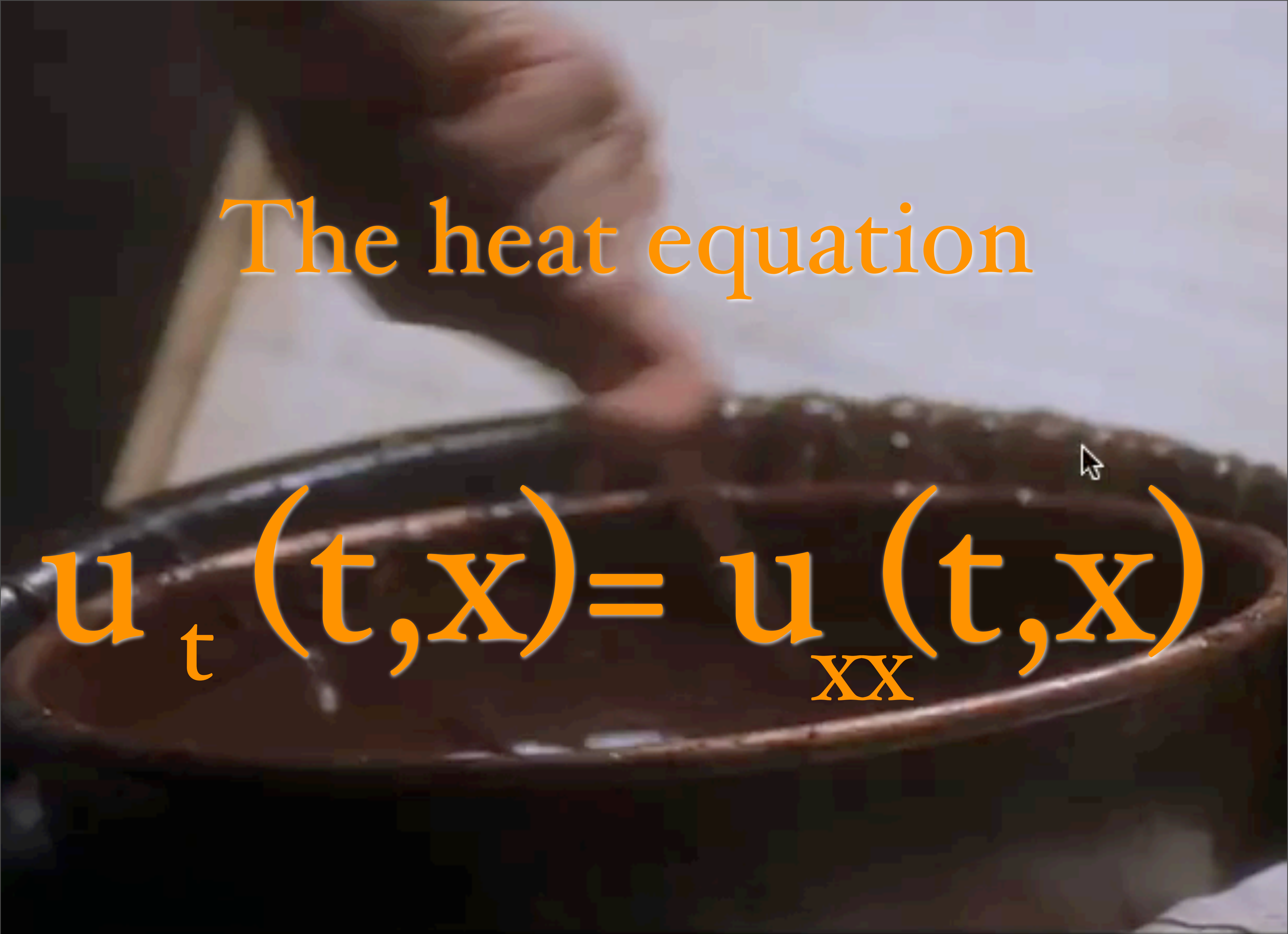
Partial differential equations

$$u_t(t,x) = u_x(t,x)$$

The transport equation

The Laplace equation

$$u_{tt}(t,x) + u_{xx}(t,x) = 0$$

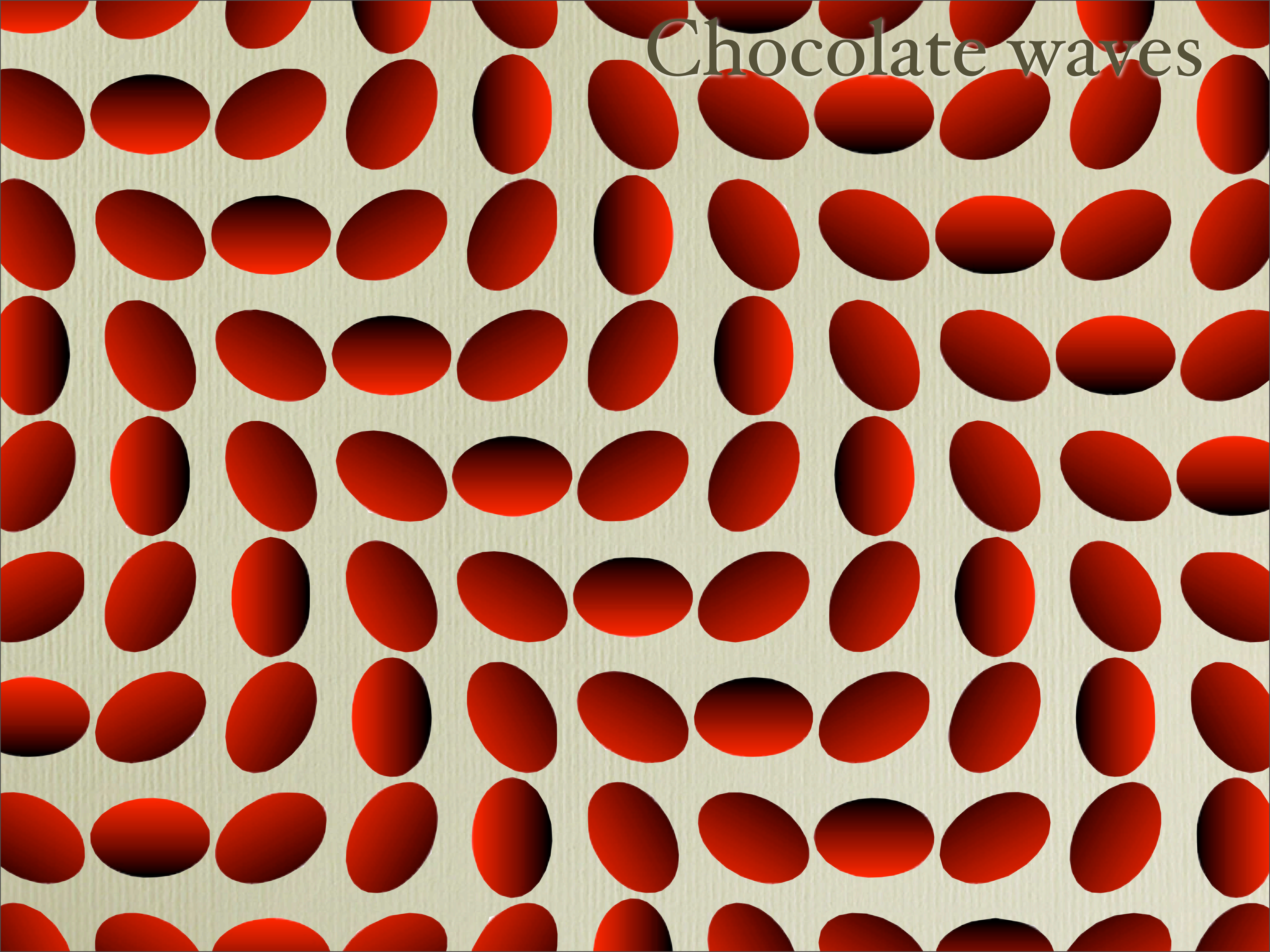


The heat equation

$$u_t(t,x) = u_{xx}(t,x)$$

The wave equation

$$u_{tt}(t,x) = u_{xx}(t,x)$$


Chocolate waves

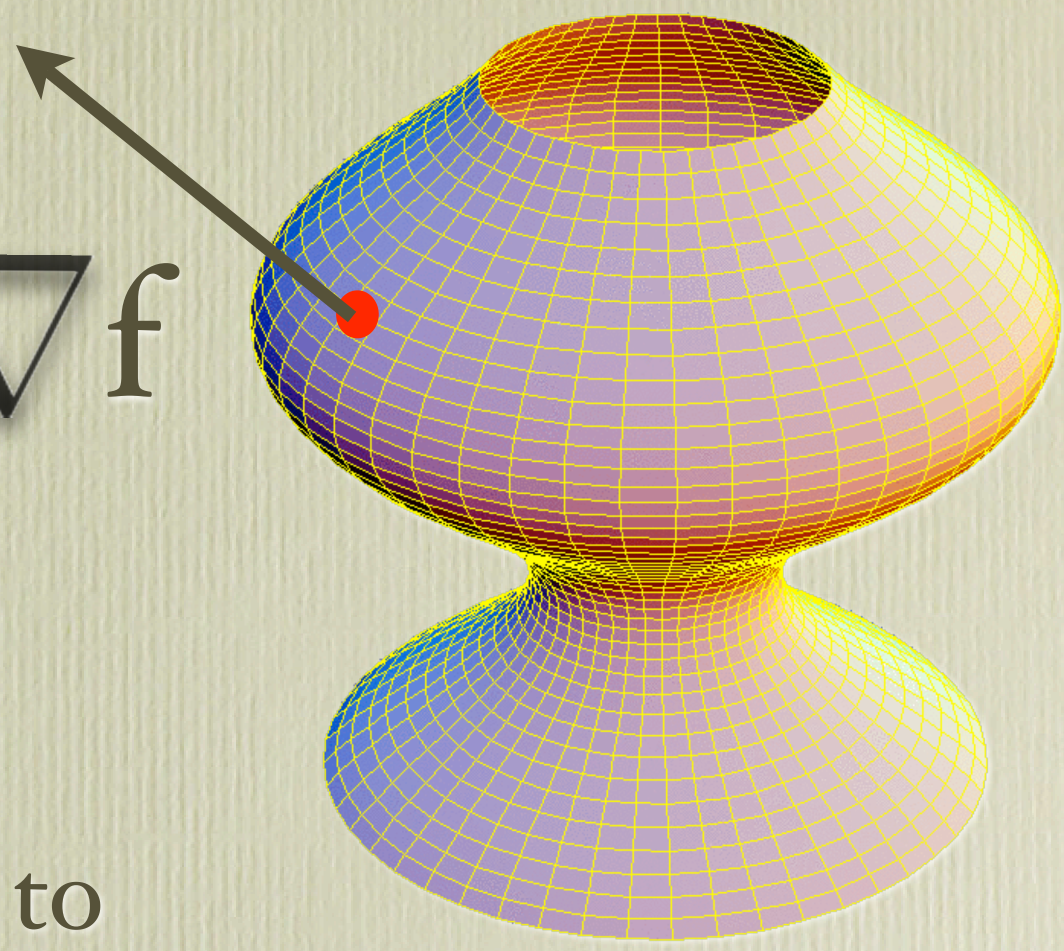
The gradient

The gradient

- Nabla ∇

$$\nabla f = < f_x, f_y, f_z >$$

Important Fact

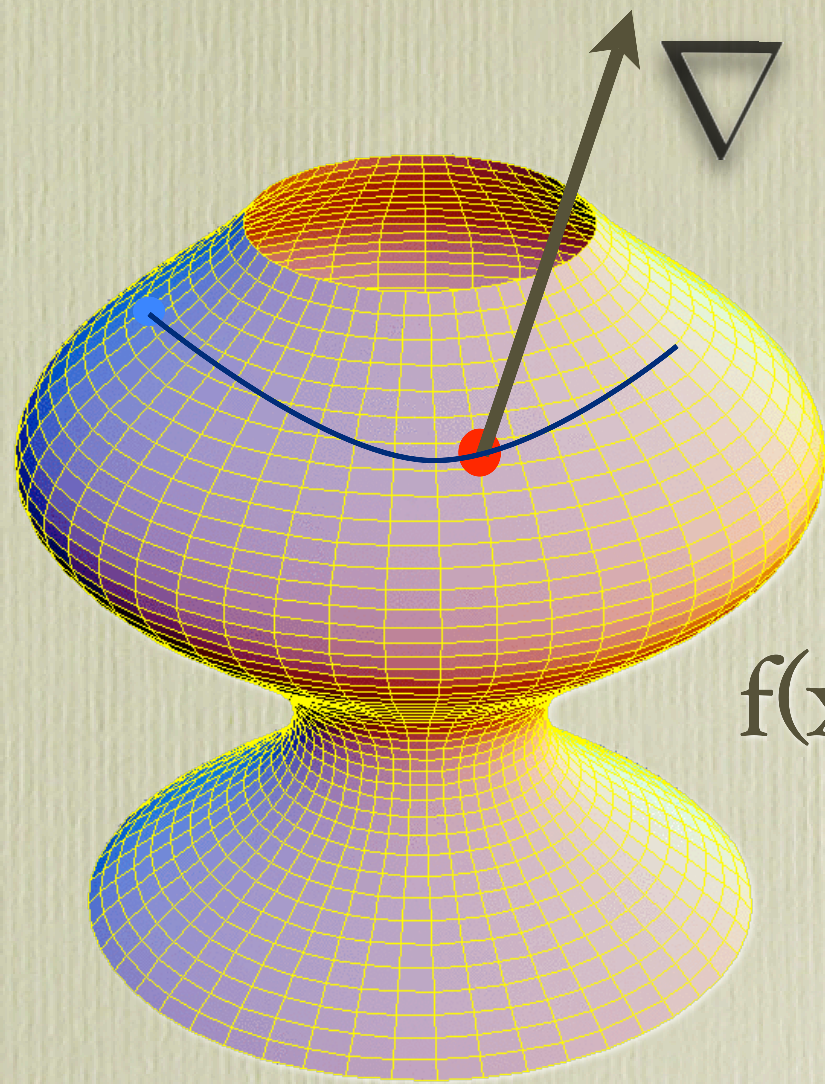


∇f

- The gradient vector is perpendicular to the level set.

$$f(x,y,z) = c$$

Proof

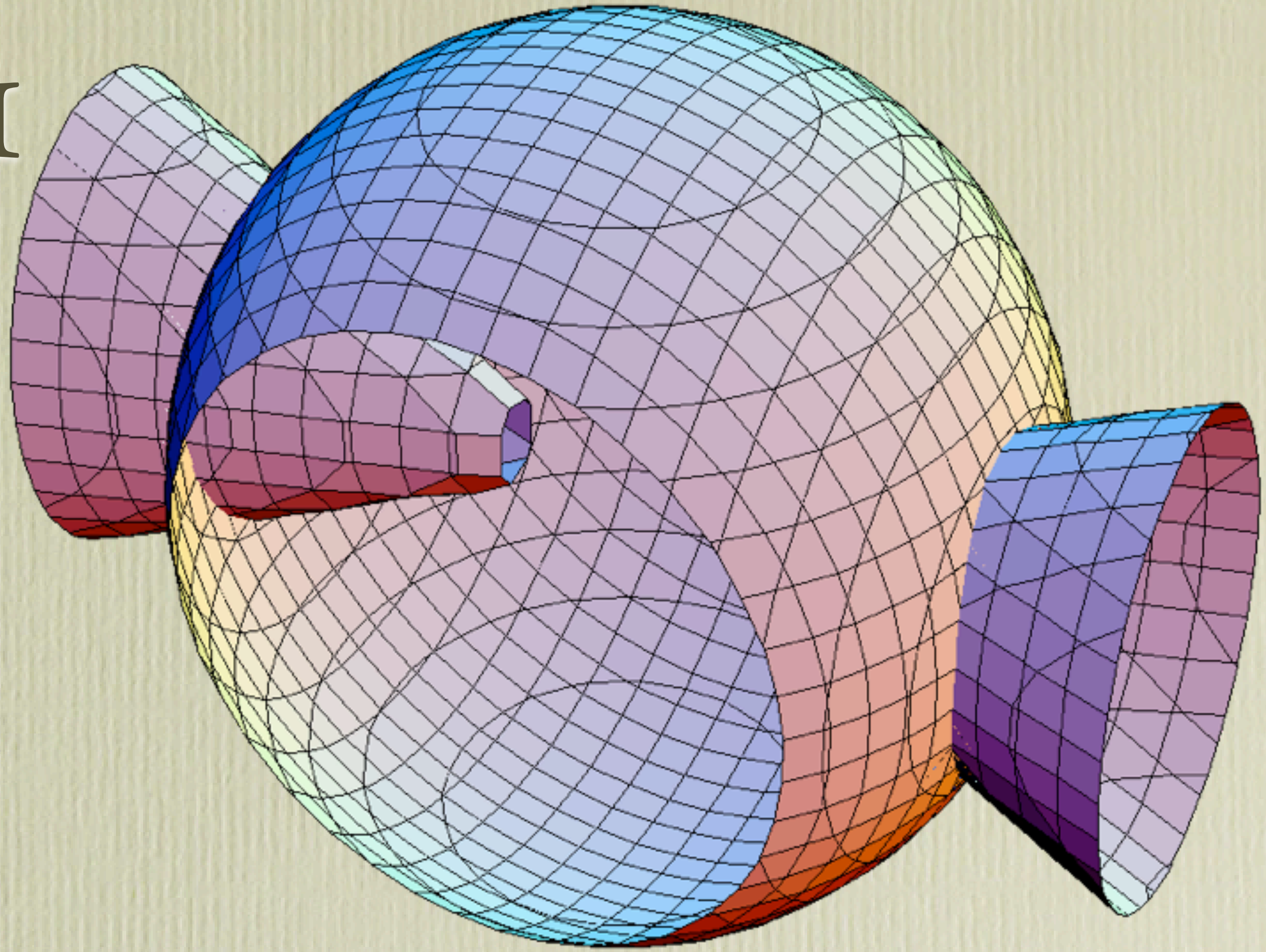


$f(x,y,z)=c$

$f(r(t))=\text{const}$

$0 = dt\ f(r(t)) = \nabla f(r(t))\ .\ r'(t)$

Problem 1



- Show that the sphere $x^2 + y^2 + z^2 = r^2$ and the elliptic cone $y^2 = x^2 + z^2$ are perpendicular at every point of their intersection.

Problem 2

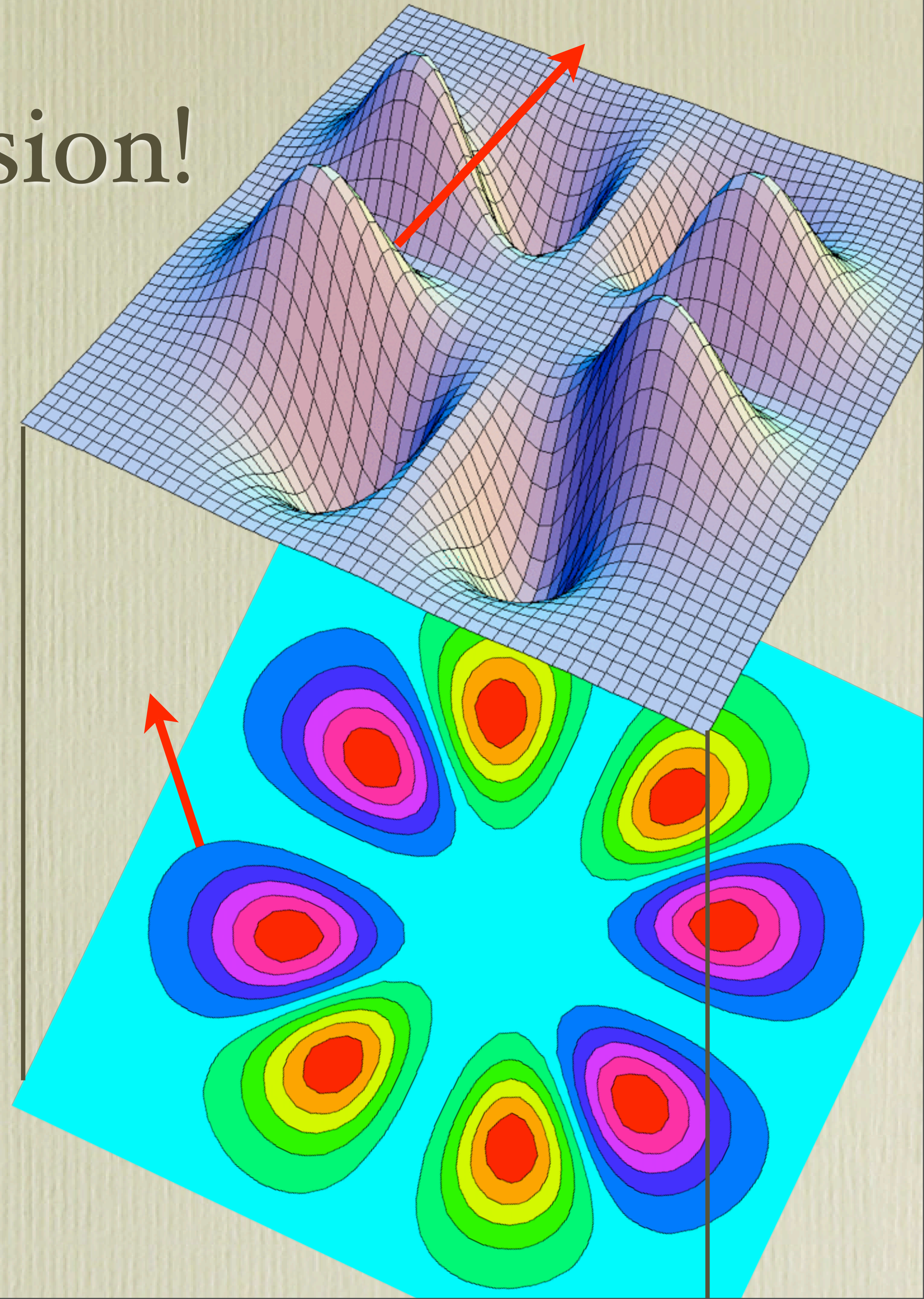
Find the normal vector to the curve

$$2xy + e^{x-1}\cos(y) = 1$$

at (1,0).

Mind the dimension!

- Watch out for the dimensions.
- The gradient of a function of 2 variables is a vector in the plane
- The gradient of a function of 3 variables is a vector in space.




Surprise
aux noix

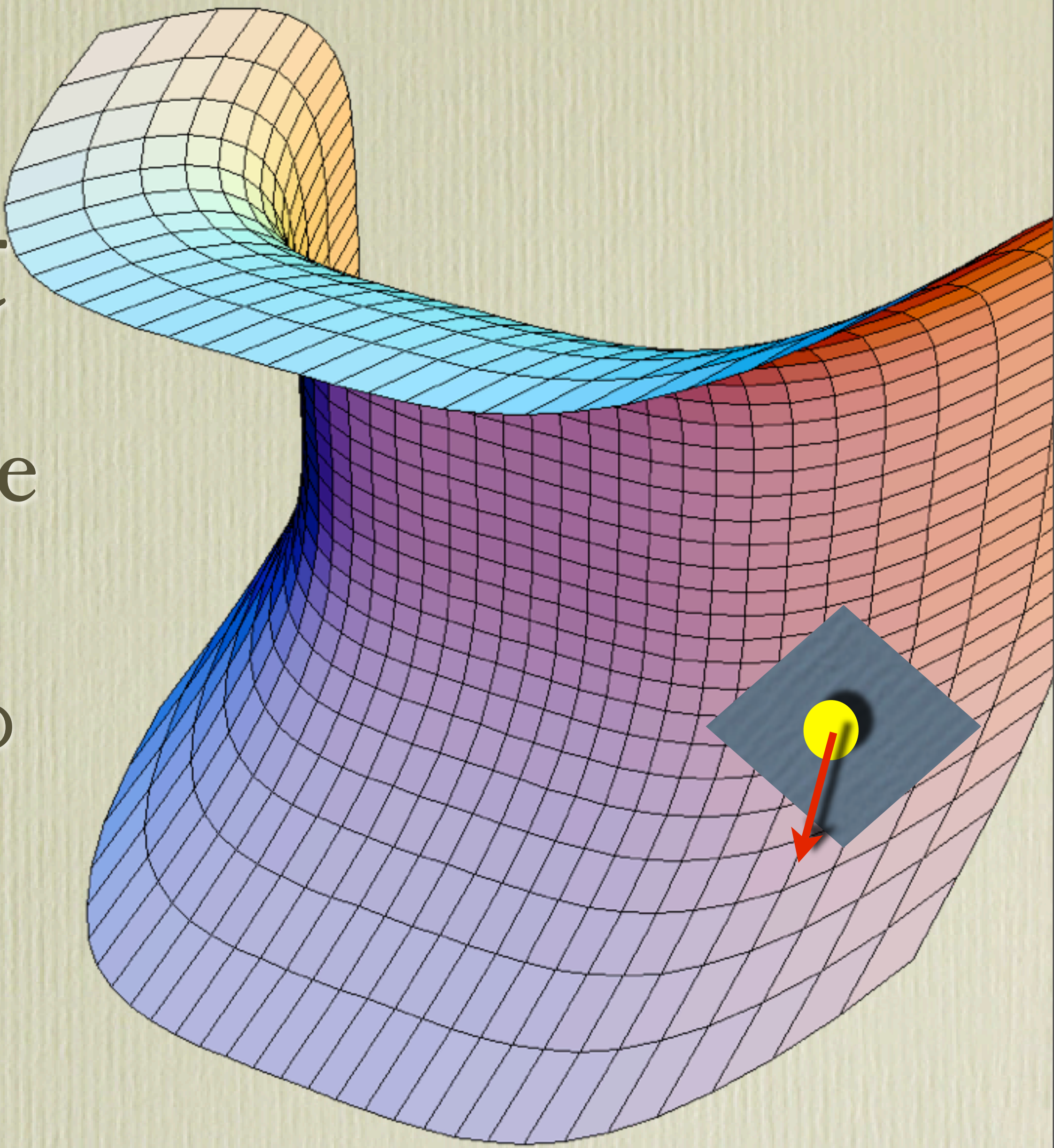
Tangent spaces

Problem 3

Find the **tangent plane** to the surface

$f(x,y,z) = x^4 - 2z^4 - y = 0$

at the point (1,0,1)

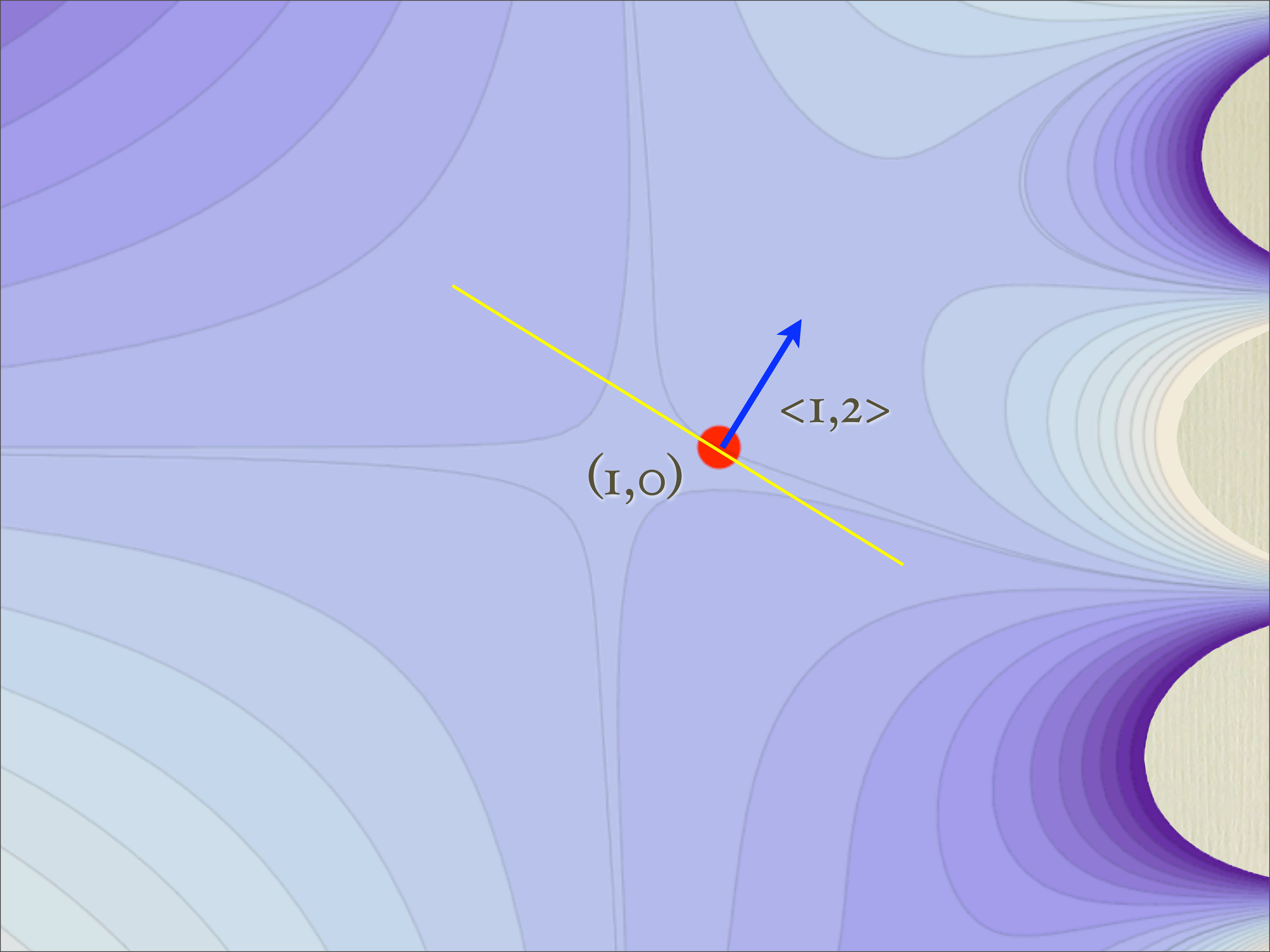


Tangent line

Find the tangent line to the curve

$$2xy + e^{x-1}\cos(y) = 1$$

at (1,0).


<1,2>
(1,0)

Linearization, estimation

Linearization, estimation

$$L(x,y) = f(a,b) + \nabla f(a,b) \cdot (x-a, y-b)$$

this function is close to f(x,y) near (a,b).

Problem 5

Can you estimate

$f(0.9999999, 0.9999, 0.99999)$ for

$f(x,y,z) = x^{-1} + 2y^{-1} + 3z^{-1}$

My main man:

Can you check this out without your brain blowing up?

Ali G Science

Ali G Estimation

Can you estimate

$f(0.9999999, 0.9999, 0.99999)$ for

$f(x,y,z) = x^{-1} + 2y^{-1} + 3z^{-1}$

without your brain blowing up?

Quiz coming up!

There will be one question

The first person who shouts the correct answer and can provide a good explanation wins a Swiss chocolate box.

Olivers Mac Mini

Warmup

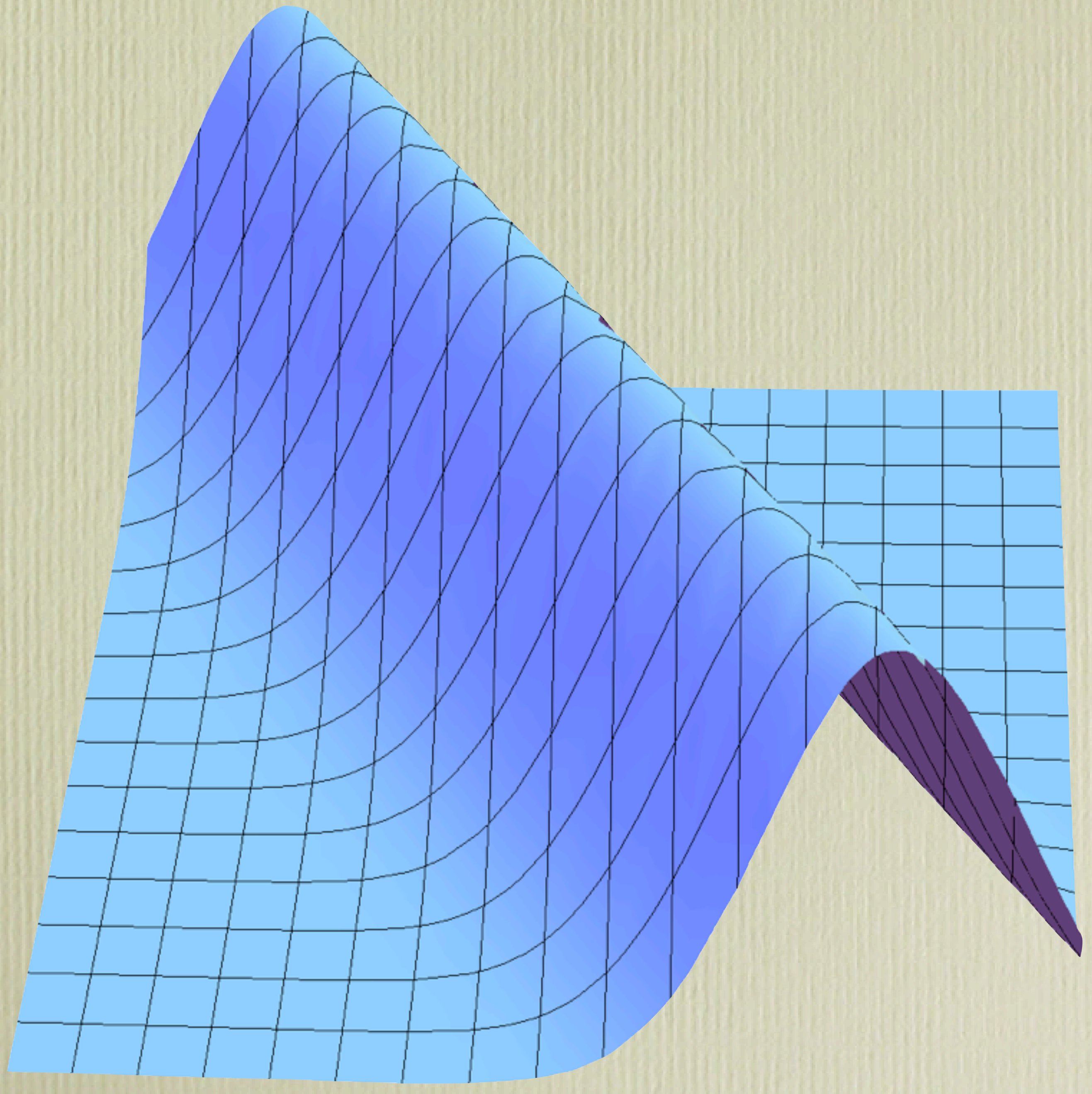
Problem 6

You know that f(x,y) satisfies the transport PDE

$$f_x = f_y$$

What can you say about the critical points of f?

The function f(x,y)



Ready

Steady

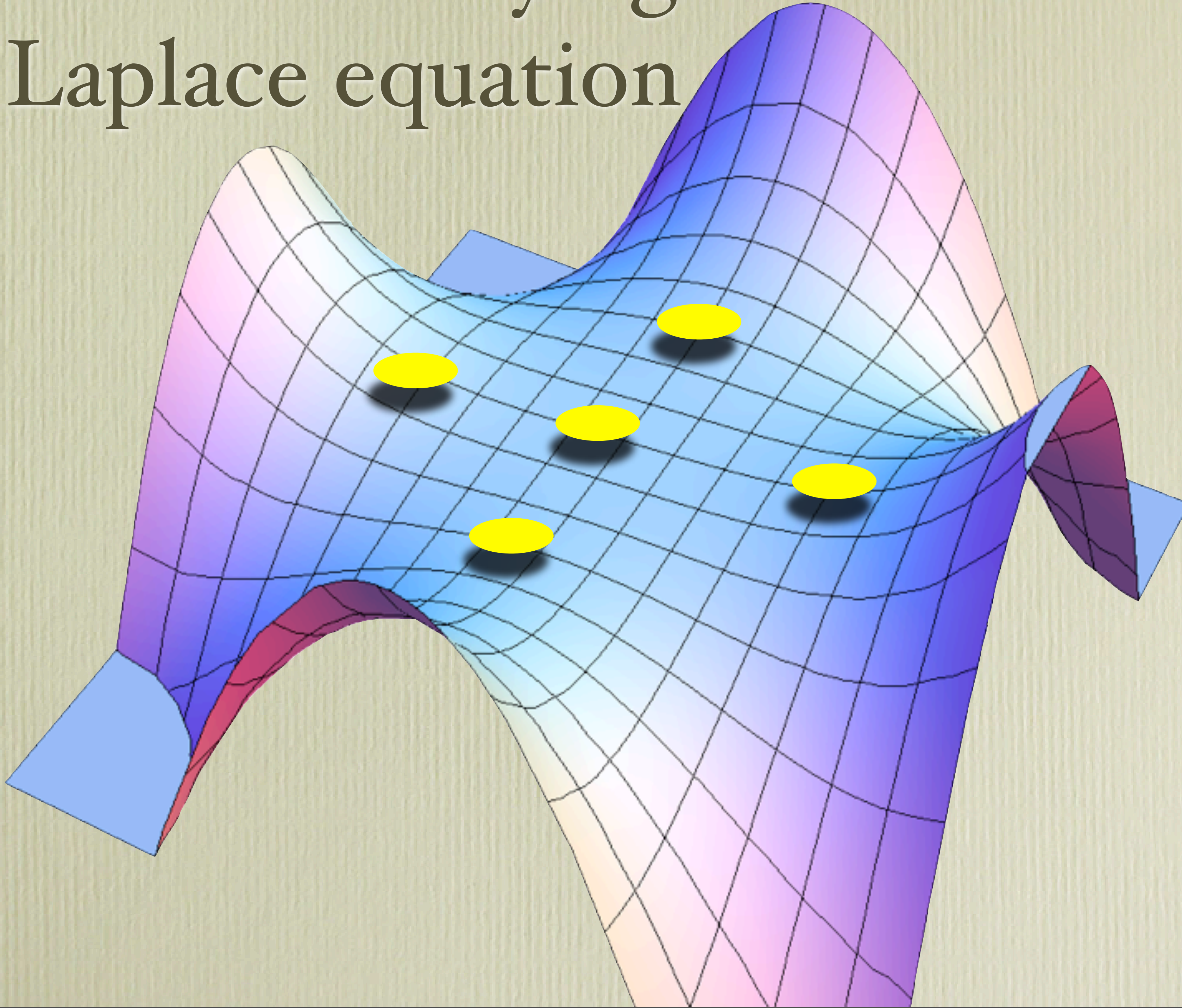
Go

You know that f(x,y) satisfies the Laplace equation

$$f_{xx} + f_{yy} = 0$$

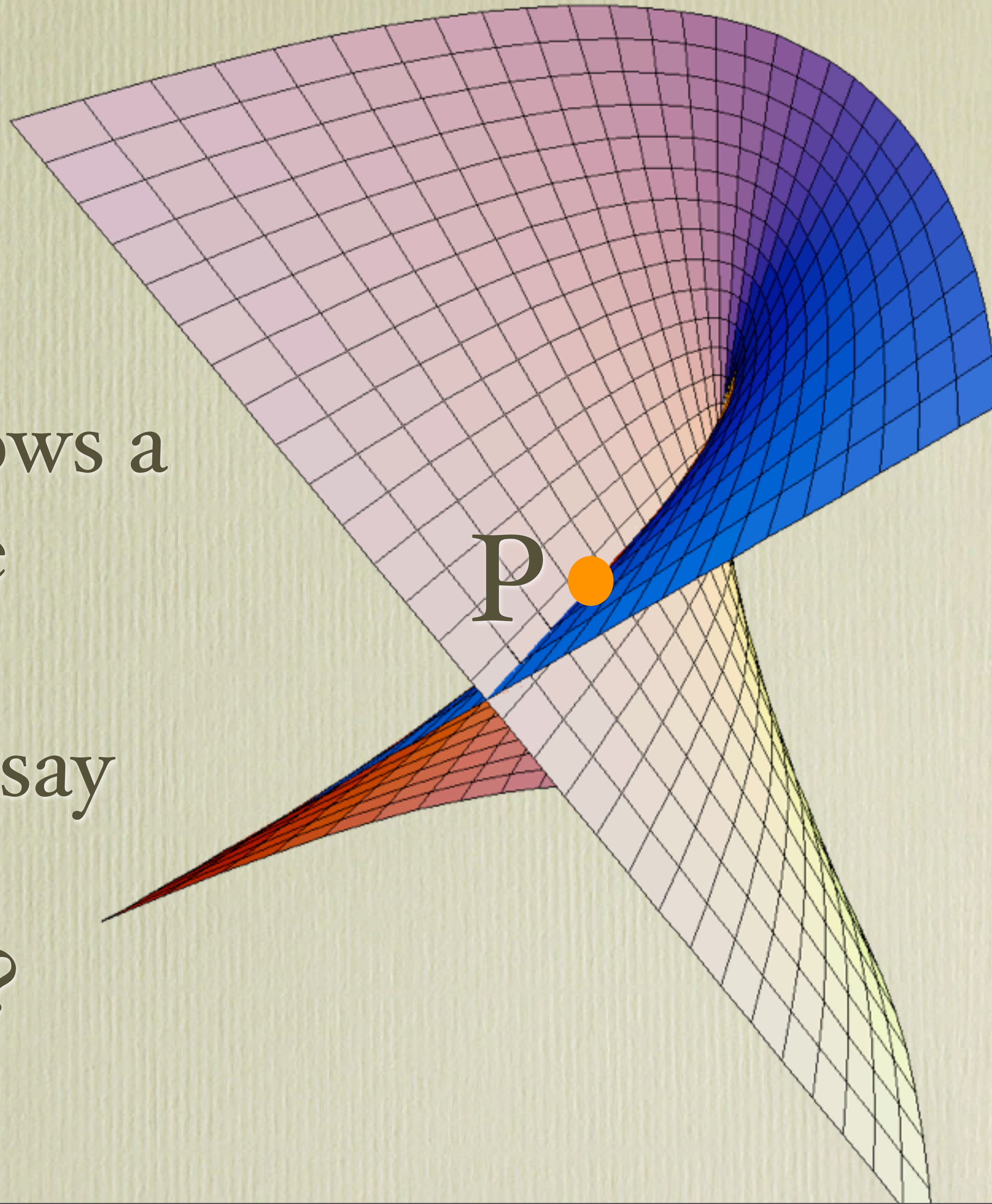
What can you say about the critical points of f(x,y) for which D is different from zero?

A function satisfying the Laplace equation



Problem:

The picture shows a level surface $f(x,y,z)=c$.
What can you say about the point P ?



The chain rule

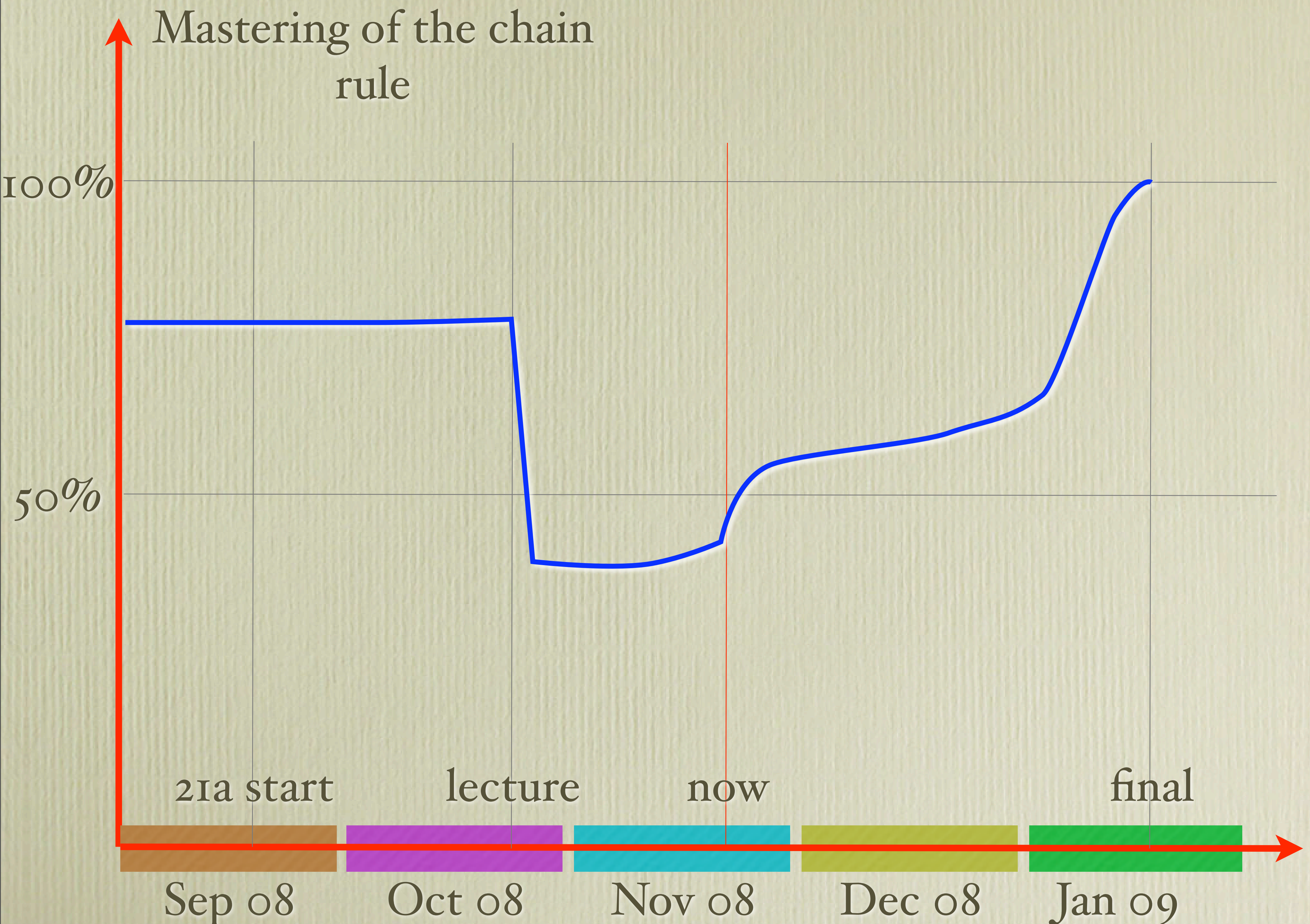
The chain rule

$$\frac{d}{dt} f(x(t),y(t)) =$$

$$f_x(x(t),y(t))\, x'(t) \quad +$$

$$f_y(x(t),y(t))\, y'(t)$$

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$



Mastering of the chain rule
100%
50%
21a start
lecture
now
final
Sep 08
Oct 08
Nov 08
Dec 08
Jan 09

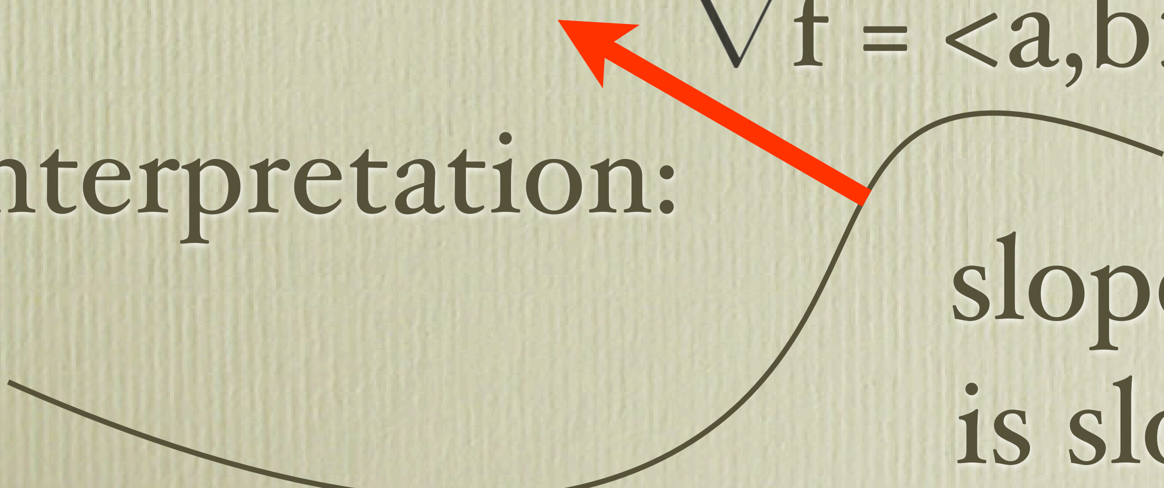
Implicit differentiation 2D

$$\frac{d}{dt} f(t,g(t)) = f_x(t,g(t))\, 1 + f_y(t,g(t))\, g'(t)$$

$$g'(t) = -f_x(t,g(t))/f_y(t,g(t))$$

$$\nabla f = \langle a,b \rangle$$

interpretation:



slope: b/a of gradient
is slope -a/b of graph

Problem:

$$\sin(x\,y) + x\,y = 0$$

Find $y'(x)$ at $x=1$.

Additionally: Find the tangent line.

Implicit differentiation 3D

$$\frac{d}{dx} f(x,y(x,y),z(x,y)) = f_x(x,y,z)\, 1 + f_y(x,y,z)\, 1 + f_z(x,y,z)\, z_x(x,y)$$

$$z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$$

similarly

$$z_y(x,y) = -f_y(x,y,z)/f_z(x,y,z)$$

Directional Derivative

$$D_{\vec{v}} f = \nabla f(x,y) \cdot \vec{v}$$

Rate of change of f in the direction $\vec{v}$. The vector $\vec{v}$ is a unit vector.

The Rollerman Problem

- Jean-Yves Blondeau aka Rollerman

Problem 10

The mountain has the height $x^4 + y^2$

Jean-Yves drives along the path $\vec{r}(t) = \langle 1-t, -\sin(t) \rangle$.

What actual slope does he experience at time $t=0$?

At time t=0, we have

$\vec{r}(0) = \langle 1, 0 \rangle$

and

$\vec{r}\,'(0) = \langle -1, -1 \rangle$.

The gradient vector at the point(x,y) is $\langle 4x^3, 2y^2 \rangle$.

At (1,0),it is $= \langle 4,0 \rangle$

The slope is the directional derivative in the direction $\langle -1,-1 \rangle/\sqrt{2}$.

Which is $-4/\sqrt{2}$

Full body rollerblading also works in town:

Steepest Ascent

$$\frac{\nabla f(x,y)}{|\nabla f(x,y)|}$$

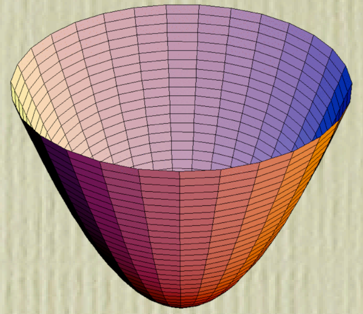
is the direction in which f increases most.


Here
is
an
Illustration

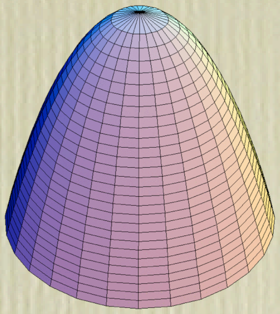
Alcohol & calculus don't mix. Never drink & derive.

Sweet extrema

Extrema without constraints



Second derivative test

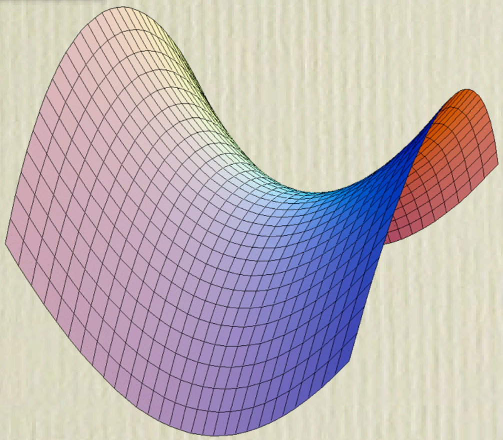


Second derivative test

$D > 0, \quad f_{xx} > 0$	Min
$D > 0, \quad f_{xx} < 0$	Max
$D < 0,$	Saddle

$$D = f_{xx} \cdot f_{yy} - f_{xy}^2$$

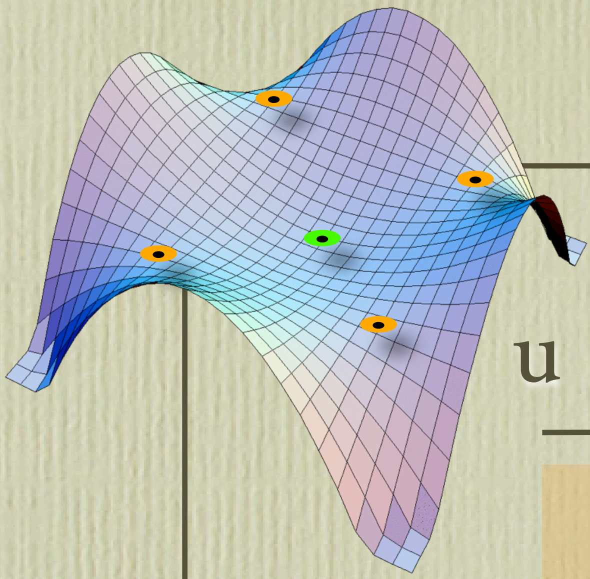
D Discriminant at the critical point (x,y)



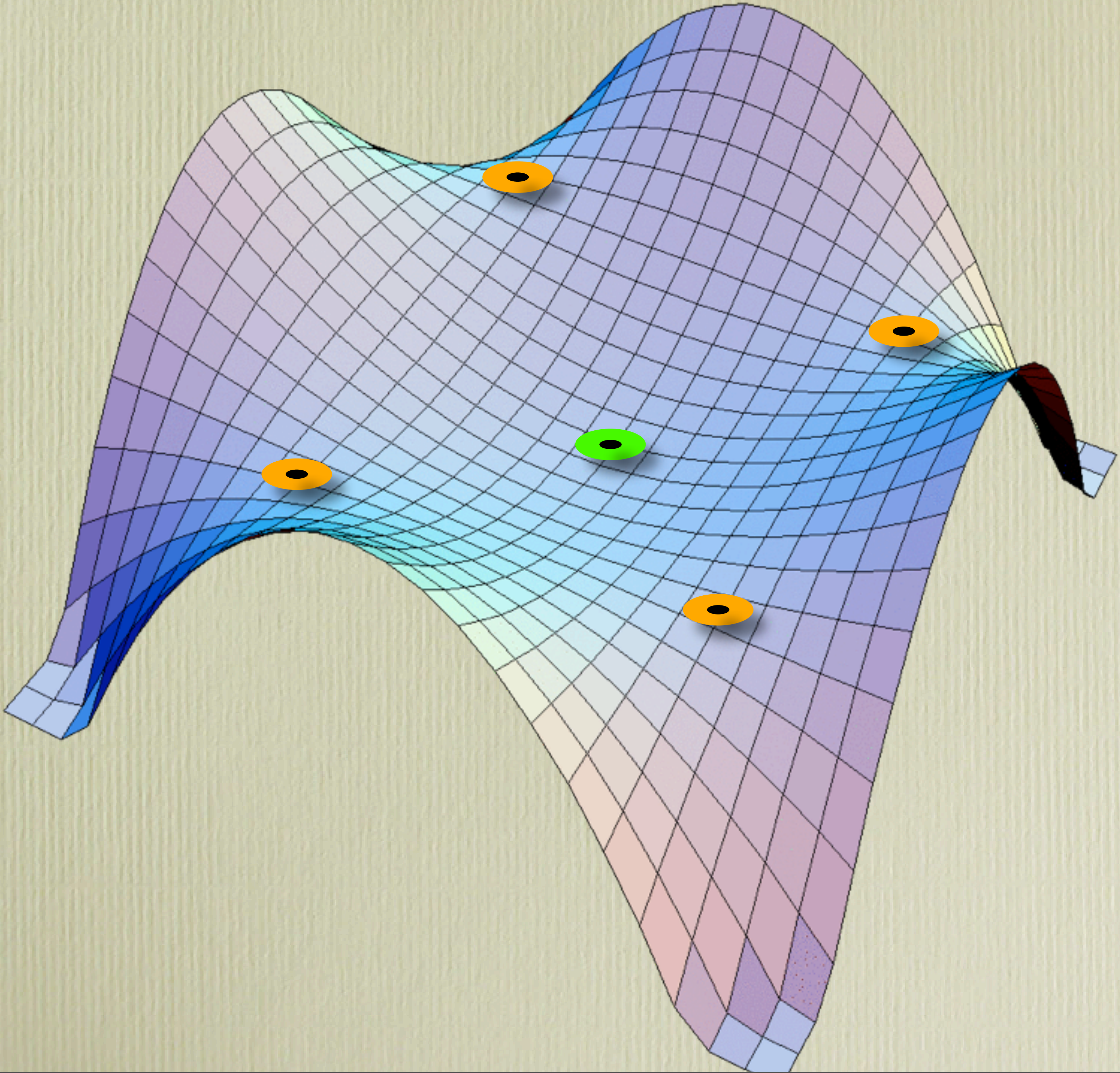
Problem 11

Find the extrema of the function

$$f(x,y) = x^2 + y^2 - x^2 y^2$$



u	v	D	f_{uu}	Type	f
-1	-1	-16	0	saddle	1
-1	1	-16	0	saddle	1
0	0	4	2	min	0
1	-1	-16	0	saddle	1
1	1	-16	0	saddle	1

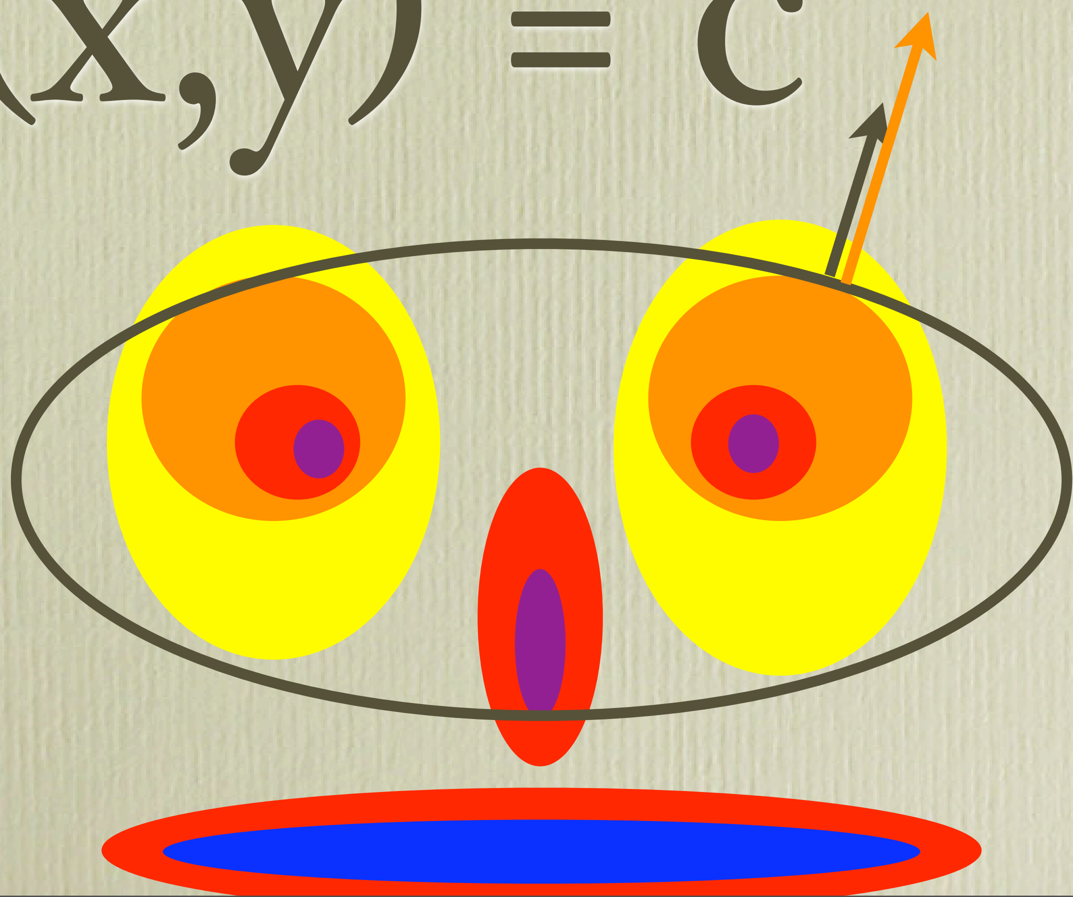


Extrema with constraints

Lagrange Equations

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

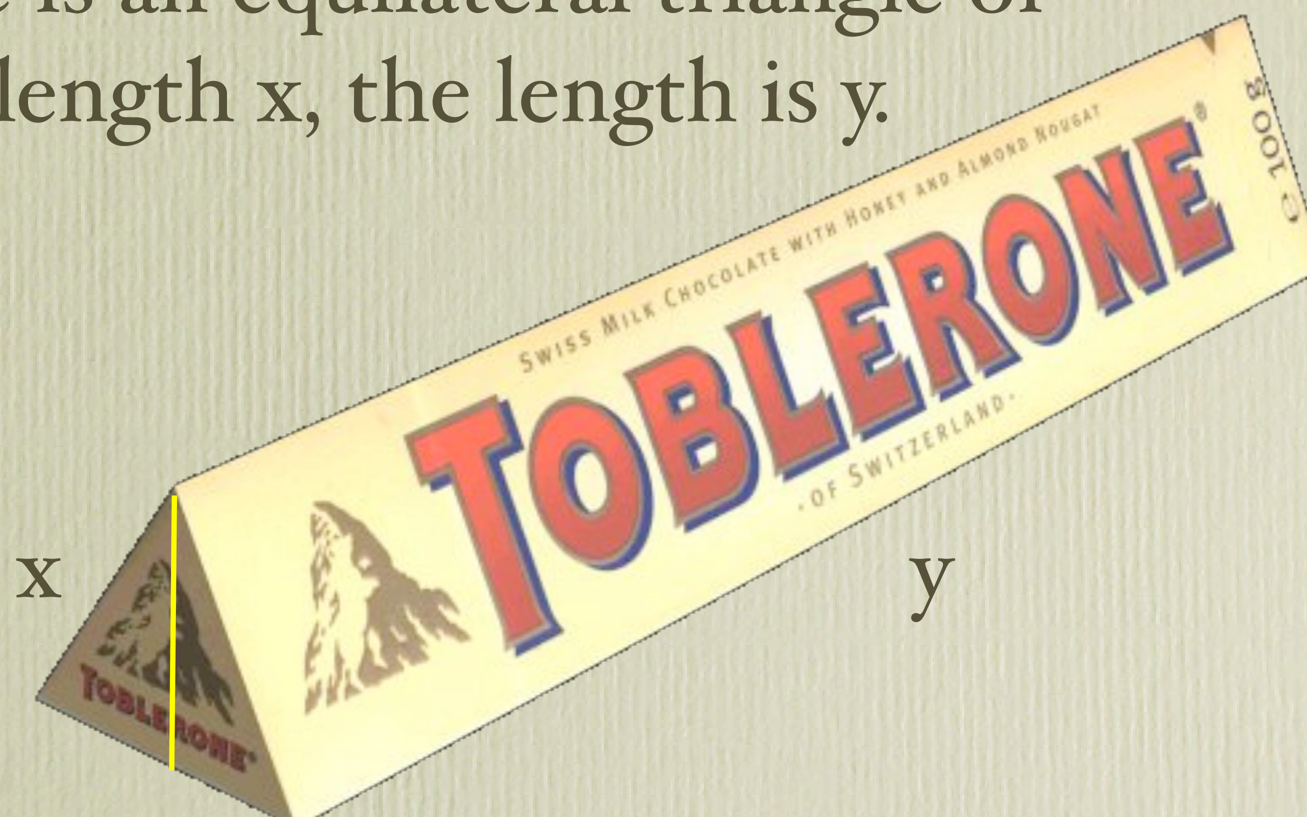
$$g(x,y) = c$$



Toblerone Problem

Toblerone Problem 12

Find the toblerone chocolate shape which has maximal volume if the surface area is constant 1.
The base is an equilateral triangle of side length x, the length is y.



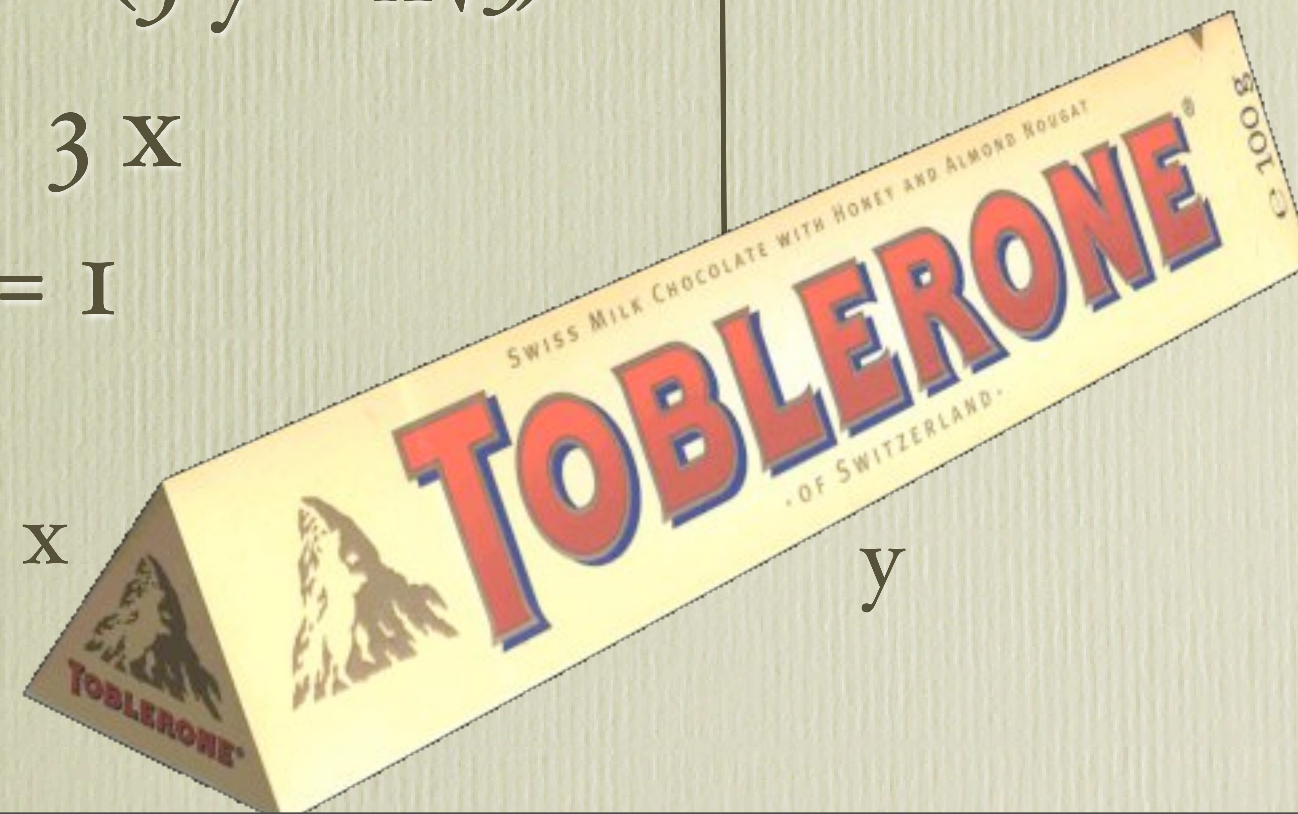
$$g = 3xy + x^2\frac{\sqrt{3}}{2} = 1$$

$$f = x^2 y \frac{\sqrt{3}}{4}$$

1. $xy\frac{\sqrt{3}}{2} = \lambda\,(3y + x\sqrt{3})$
2. $x^2\frac{\sqrt{3}}{4} = \lambda\, 3x$
3. $g = 1$

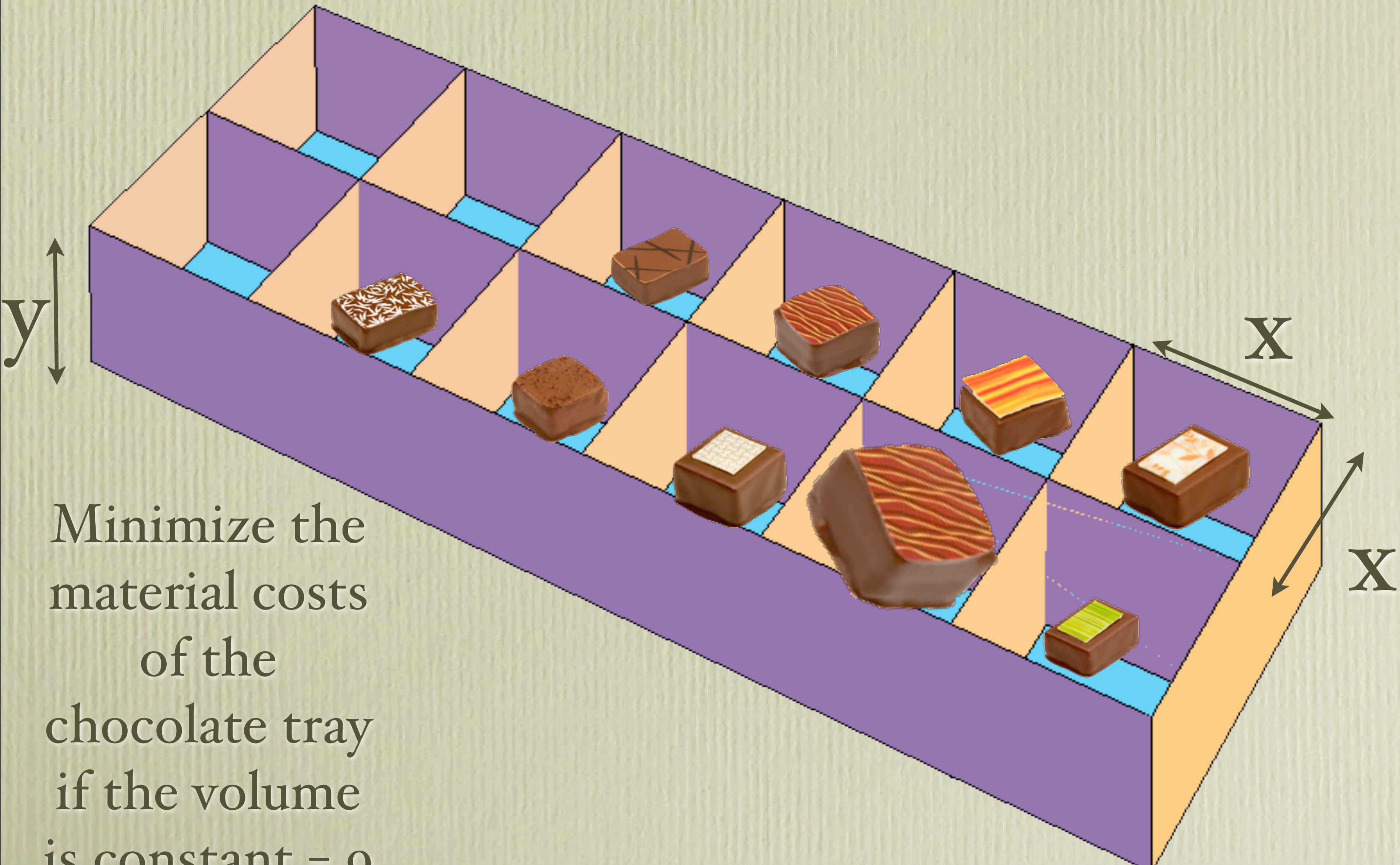
$\dfrac{1}{2}$: $x/y = \sqrt{3}$

plug into 3



Exams are like box of chocolates:

Chocolate Tray Problem



Minimize the material costs of the chocolate tray if the volume is constant = 9

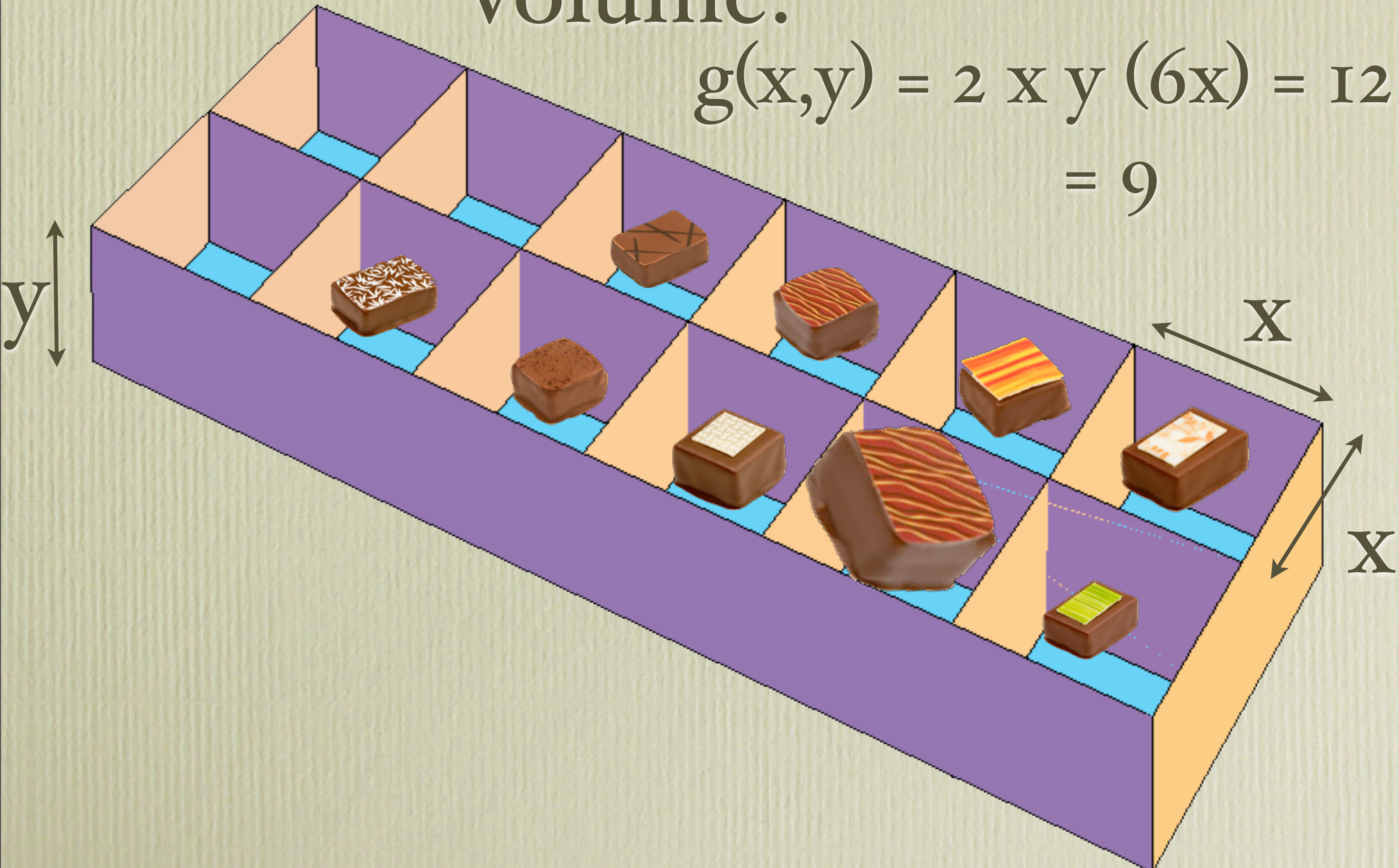
Area: $f(x,y) = (2x)(6x) + 3(6xy) + 7(2xy)$

$= 12x^2 + 32xy$

Volume:

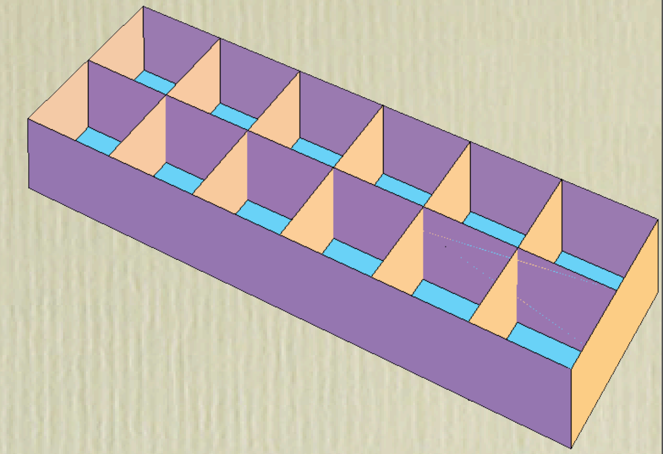
$g(x,y) = 2xy(6x) = 12x^2y$

$= 9$



Area: $f(x,y) = 12\,x^2 + 32\,x\,y$

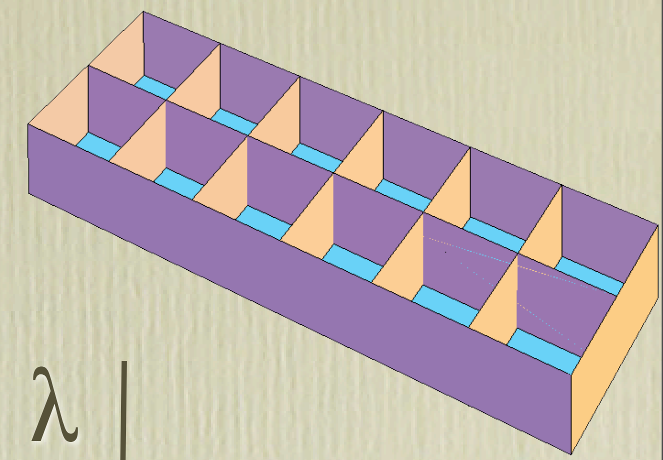
Volume: $g(x,y) = 12\,x^2 y = 9$



Lagrange equations:

$$\left|\begin{array}{l} 24\,x + 32\,y = 24\,x\,y \quad \lambda \\ 32\,x = 12\,x^2 \quad \lambda \\ 12\,x^2 y = 9 \end{array}\right|$$

Solving the Lagrange equations:



$$\textcircled{1} \quad 24\,x + 32\,y = 24\,x\,y\,\lambda$$

$$\textcircled{2} \quad 32\,x = 12\,x^2\,\lambda$$

$$\textcircled{3} \quad 12\,x^2 y = 9$$

$$\textcircled{1} / \textcircled{2} \qquad \frac{24}{32} + \frac{y}{x} = \frac{2y}{x}$$

$$4y = 3\,x$$

$$9\,x^3 = 9$$

$x=1$,

$y=3/4$

Global extrema

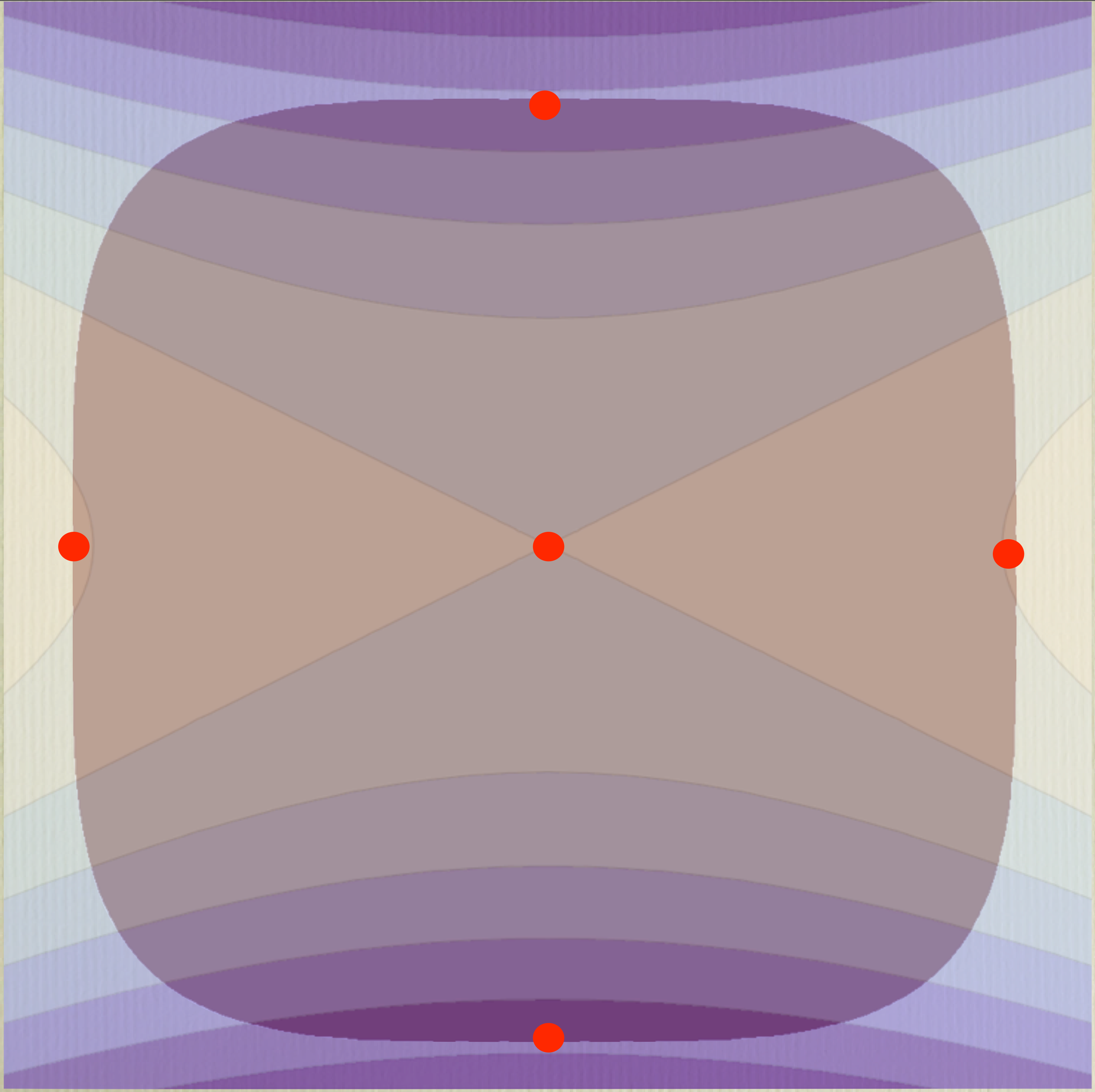
Problem:

Sugar concentration on the top of a cake is

$$f(x,y) = x^2 - 4y^2$$

The cake surface is the domain $x^4 + y^4 \leq 1$

Where is the sugar concentration maximal?



Problem :

Where is the global minimum of

$$f(x,y) = x^2 - 2x + y^2 - 4y + 1$$

on $y \geq x^2$

Extrema:

$$2x-2 = 0$$
$$2y-4 = 0$$

$x=1$
$y=2$

Extrema with constraints:

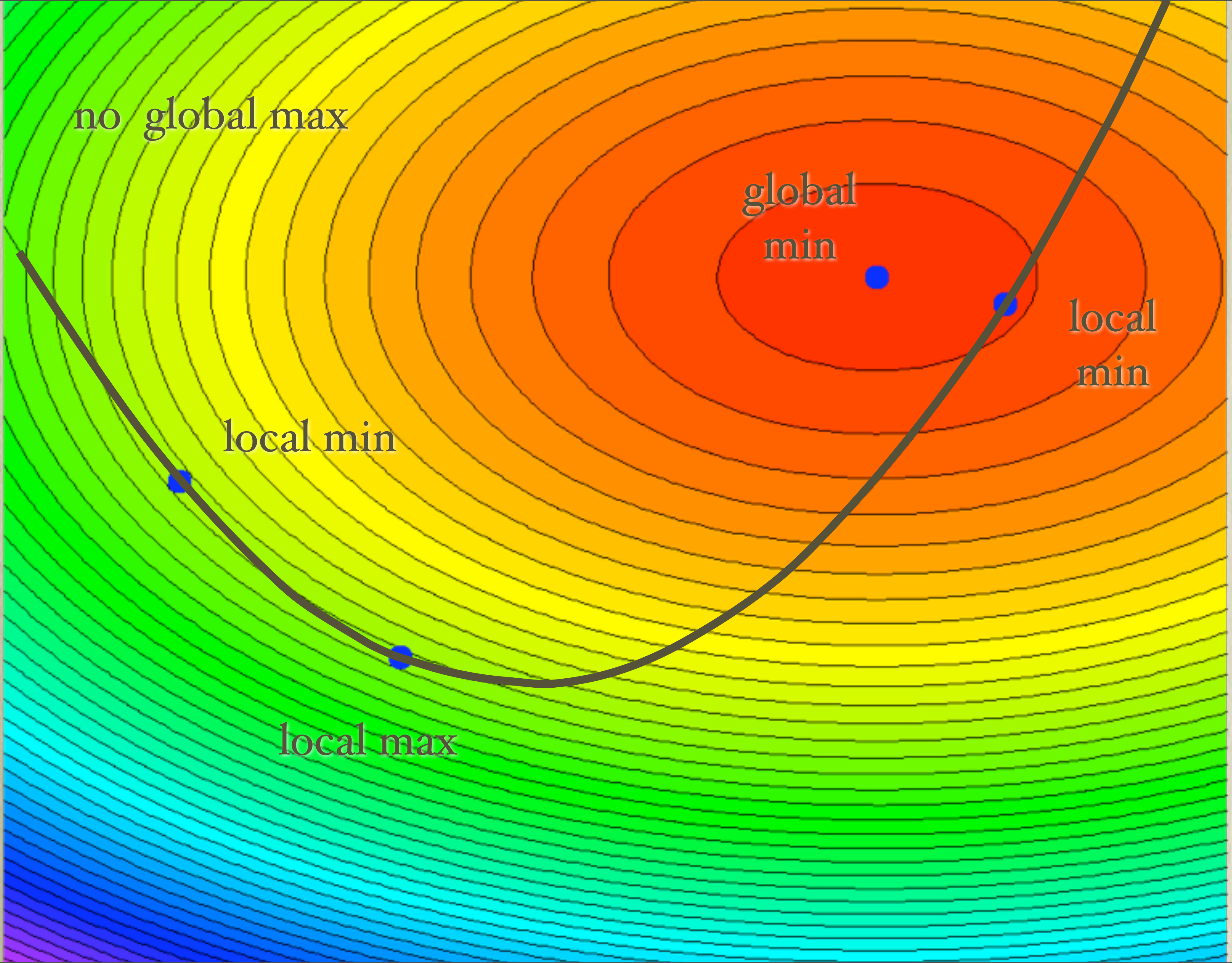
$$\begin{array}{ll} (1) & 2x-2 = -2x\lambda \\ (2) & 2y-4 = \lambda \\ (3) & y - x^2 = 0 \end{array}$$

$(1)/(2)$

$$\frac{2x-2}{2y-4} = -2x \qquad y - x^2 = 0$$

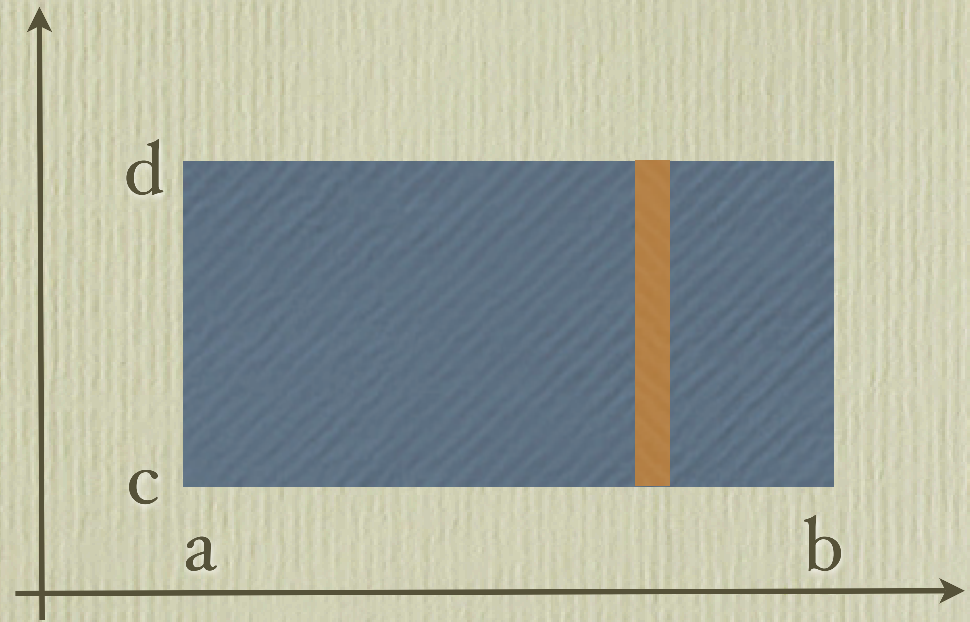
solutions are:

$(-1,1)$, $((1\pm\sqrt{3})/2,(2\pm\sqrt{3})/2)$


no global max
global
min
local
min
local min
local max

Integration of functions of two variables.

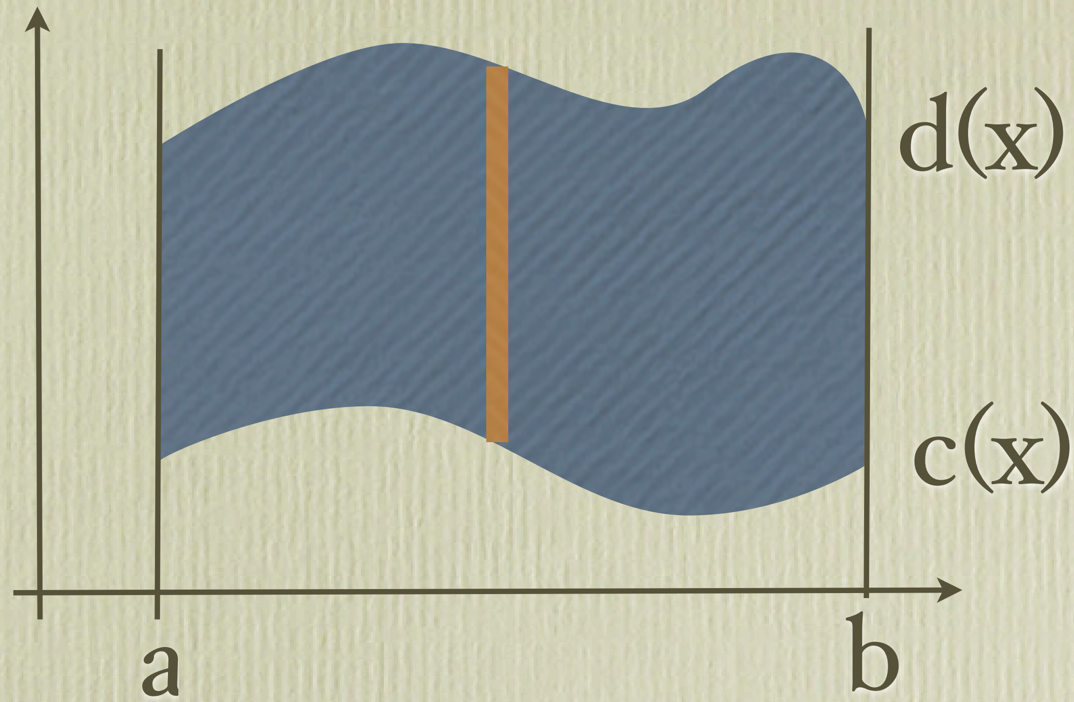
Integration in two dimensions



$$\int_a^b \int_c^d f(x,y)\, dy\, dx$$

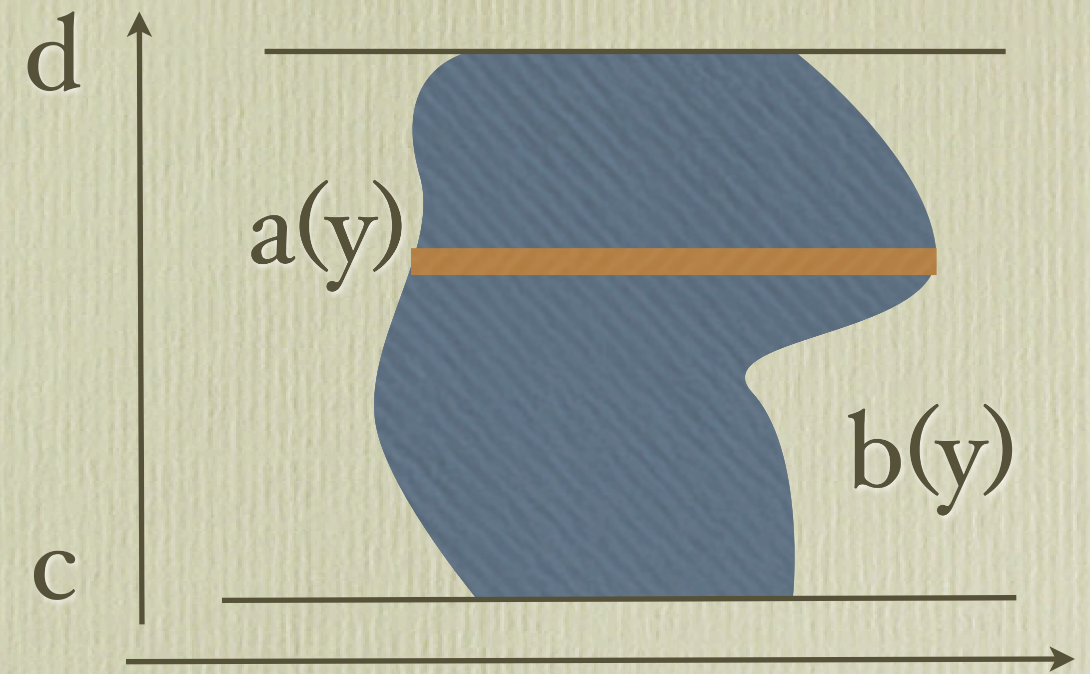
$$\int_c^d \int_a^b f(x,y)\, dx\, dy$$

Type I integrals



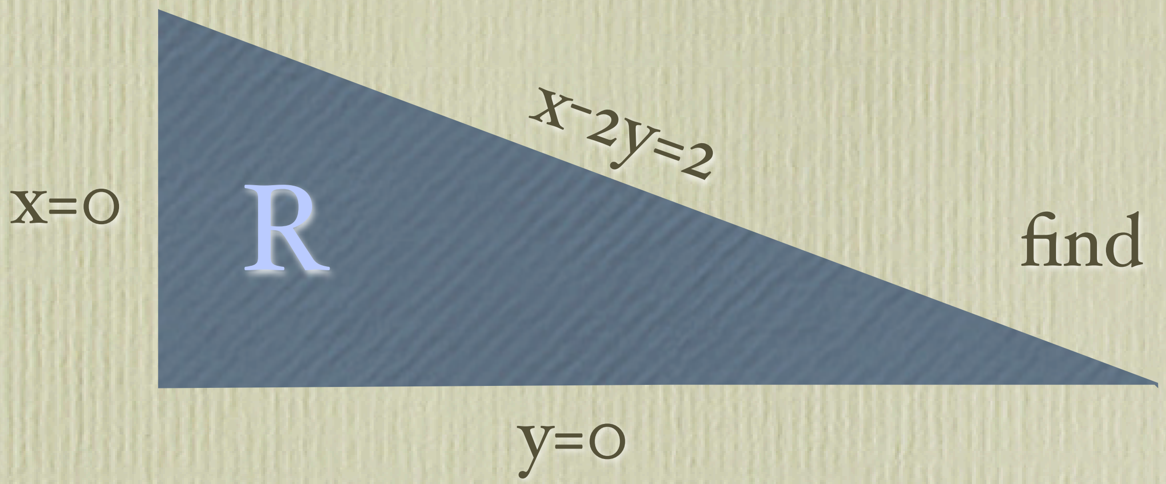
$$\int_{a}^{b}\int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

Type II integrals



$$\int_{c}^{d}\int_{a(y)}^{b(y)} f(x,y)\,dx\,dy$$

Order of integration



find $\iint_R e^y/(y+1)\, dA$

As a type I region $\int_0^2 \int_0^{x/2-1} e^y/(y+1)\, dy\, dx$

As a type II region $\int_0^1 \int_0^{2+2y} e^y/(y+1)\, dx\, dy$

Setting up integrals

Let R be the region bounded by

$$y = x^2 - 1$$

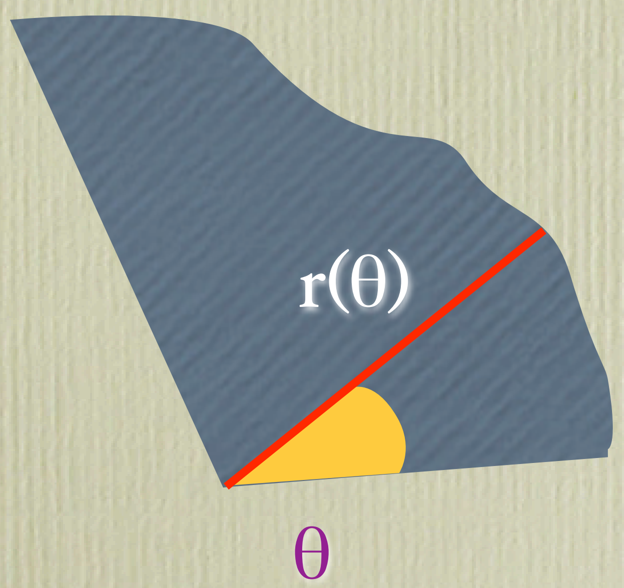
and the unit circle $x^2 + y^2 = 1$

$$\iint_R f(x,y)\, dA$$

Polar integration

$$\iint f(x,y)\, dx\, dy \quad = \iint f(r \cos(\theta), r \sin(\theta)\, r\, dr\, d\theta$$

Typical region

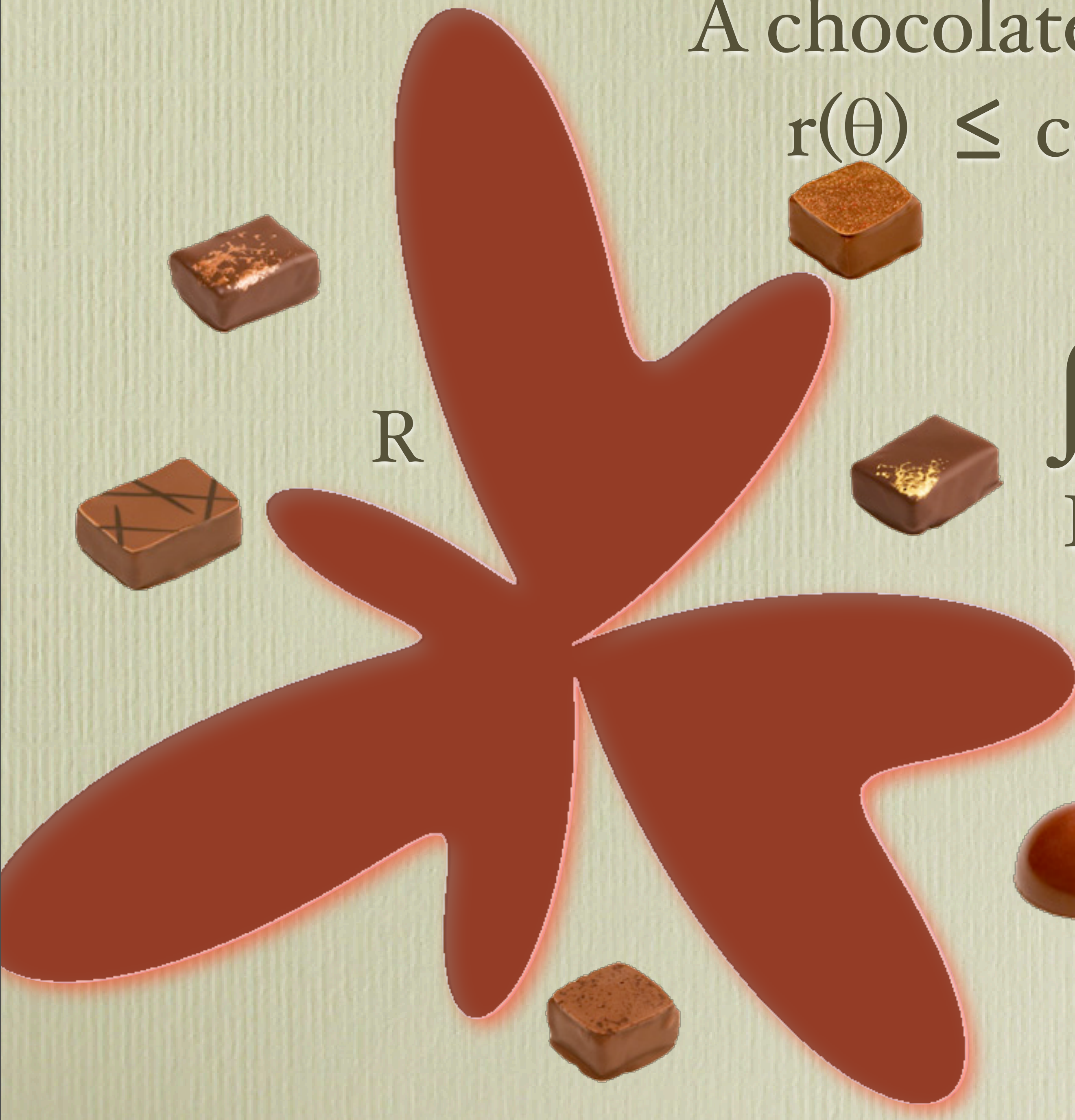


$$\int_a^b \int_0^{r(\theta)}$$

Chocolate cookie problem

A chocolate cookie is satisfies

$$r(\theta) \leq \cos(7\,\theta) - \sin(3\,\theta) + 2$$



$$\iint_R 1\,dx\,dy$$

Advise

- make a picture - in any case!
- consider change order of integration
- use of other coordinates?