

Name:

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|--------------------------|
| MWF 9 Chen-Yu Chi |
| MWF 10 Janet Chen |
| MWF 10 Sug Woo Shin |
| MWF 10 Jay Pottharst |
| MWF 11 Oliver Knill |
| MWF 11 Kai-Wen Lan |
| MWF 12 Valentino Tosatti |
| TTH 10 Gerald Sacks |
| TTH 10 Ilia Zharkov |
| TTH 11:30 David Harvey |
| TTH 11:30 Ilia Zharkov |

- Start by printing your name in the above box and check your section in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- **Show your work.** Except for problems 1-2, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|--------|--|-----|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

- 1) T F For a moving frame $(\vec{T}, \vec{N}, \vec{B})$, (remember that \vec{T} is the unit tangent vector, \vec{N} is the normal vector and \vec{B} is the binormal vector), one always has $\vec{B} \cdot (\vec{T} \times \vec{N}) = 1$.

Solution:

The three vectors have length 1 and are perpendicular to each other.

- 2) T F For any three points P, Q, R in space, $\vec{PQ} \times \vec{PR} = \vec{QP} \times \vec{RP}$

Solution:

The two vectors are switching sign on the right hand side.

- 3) T F The triangle defined by the three points $(-1, 0, 2), (-4, 2, 1), (1, -1, 2)$ has a right angle.

Solution:

It would be inefficient to compute the angles. Better is to look at the squares of the lengths of the triangle which are 14, 5 and 35. If there was a right angle, Pythagoras would apply.

- 4) T F Any two distinct non-parallel planes intersect in a line.

Solution:

Because they are non-parallel, they intersect and at least one point Q and their normal vectors have a nonzero cross product \vec{n} . The line has the parametrization $Q + t\vec{n}$.

- 5) T F $\vec{u} \times \vec{u} = 0$ implies $\vec{u} = \vec{0}$.

Solution:

The left hand side is always true. To see that it is false, take $\vec{u} = \langle 1, 0, 0 \rangle$. It is not the zero vector, but still $\vec{u} \times \vec{u} = 0$.

- 6) T F The level curves $f(x, y) = 1$ and $f(x, y) = 2$ of a smooth function f never intersect.

Solution:

If they would intersect in a point (x, y) , then f would take two values 1 and 2 at the point which is not possible.

- 7) T F For any vector \vec{v} we have $P_{\vec{i}}(P_{\vec{j}}(\vec{v})) = \vec{0}$.

Solution:

The vector $P_{\vec{j}}(\vec{v})$ is parallel to \vec{j} which is perpendicular to \vec{i} and the projection onto \vec{i} is therefore the zero vector.

- 8) T F $(\vec{j} \times \vec{i}) \times \vec{i} = \vec{k} \times (\vec{i} \times \vec{k})$

Solution:

The left vector is parallel to j , the right vector is parallel to i .

- 9) T F If a parametrized curve $\vec{r}(t)$ lies in a plane and the velocity $\vec{r}'(t)$ is never zero, then the normal vector $\vec{N}(t)$ also lies in that plane.

Solution:

This is intuitively clear.

- 10) T F The angle between $\vec{r}'(t)$ and $\vec{r}''(t)$ is always 90 degrees.

Solution:

It is true for circles, but false in general. For example, on a line, the acceleration parallel to the velocity.

- 11) T F If \vec{v}, \vec{w} are two nonzero vectors, then the projection vector $P_{\vec{w}}(\vec{v})$ can be longer than \vec{v} .

Solution:

The projection vector has length $|\vec{v} \cdot \vec{w}|/|\vec{w}|$ which has length smaller or equal to $|\vec{v}|$ (use the cos formula).

- 12)

| | |
|---|---|
| T | F |
|---|---|

 A line intersects an ellipsoid in at most 2 distinct points.

Solution:

One can see this geometrically. Here is an argument: we know it for a sphere. When stretching the picture with the sphere and line the number of intersections does not change.

- 13)

| | |
|---|---|
| T | F |
|---|---|

 For any vectors \vec{v} and \vec{w} , the formula $(\vec{v} - \vec{w}) \cdot \vec{P}_{\vec{w}}(\vec{v}) = 0$ holds.

Solution:

Take $\vec{v} = \vec{i}$ and $\vec{w} = \vec{j}$.

- 14)

| | |
|---|---|
| T | F |
|---|---|

 Let S be a plane normal to the vector \vec{n} , and let P and Q be points not on S . If $\vec{n} \cdot \vec{PQ} = 0$, then P and Q lie on the same side of S .

Solution:

The condition $\vec{n} \cdot \vec{PQ} = 0$ implies that the vector \vec{PQ} is parallel to the plane.

- 15)

| | |
|---|---|
| T | F |
|---|---|

 The vectors $\langle 2, 2, 1 \rangle$ and $\langle 1, 1, -4 \rangle$ are perpendicular.

Solution:

The dot product vanishes

- 16)

| | |
|---|---|
| T | F |
|---|---|

 $\|\vec{v} \times \vec{w}\| = \|\vec{v}\|\|\vec{w}\| \cos(\alpha)$, where α is the angle between \vec{v} and \vec{w} .

Solution:

It is sin not cos.

- 17) T F The vector $\vec{i} \times (\vec{j} \times \vec{k})$ has length 1.

Solution:

It is the zero vector.

- 18) T F The distance between the z -axis and the line $x - 1 = y = 0$ is 1.

Solution:

You can see that geometrically.

- 19) T F There is a quadric surface which both hyperbola and parabola appear as traces. Traces are intersections of the surface with the coordinate planes $x = 0$, $y = 0$, or $z = 0$.

Solution:

The hyperbolic paraboloid has that.

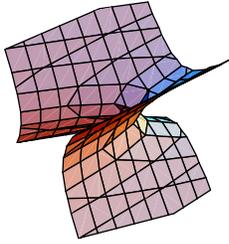
- 20) T F The equation $x^2 + y^2 - z^2 = -1$ defines a one-sheeted hyperboloid.

Solution:

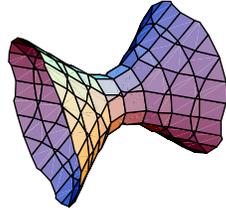
$f(x, y, z) = x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid

Problem 2) (10 points)

Match the equation with the pictures. No justifications are necessary in this problem.



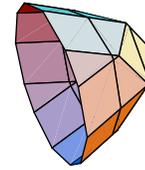
I



II

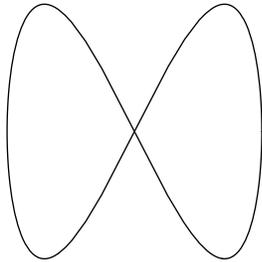


III

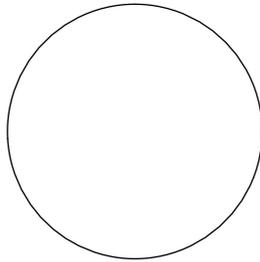


IV

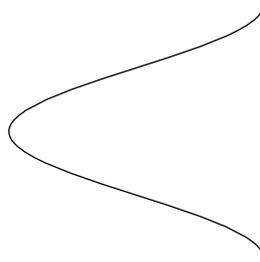
| Enter I,II,III,IV here | Equation |
|------------------------|----------------------------|
| | $x + y^2 - z^2 - 1 = 0$ |
| | $-x^2 + y^2 + z^2 - 1 = 0$ |
| | $-x^2 + y^2 + z^2 + 1 = 0$ |
| | $-x + y^2 + z^2 + 1 = 0$ |



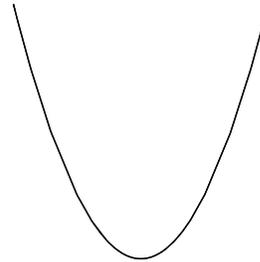
1



2

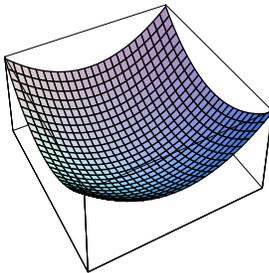


3

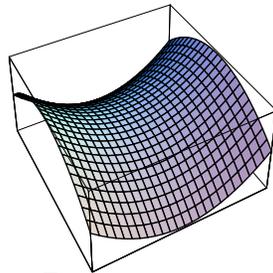


4

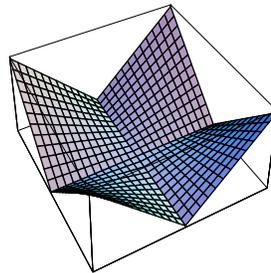
| Enter 1,2,3,4 here | Equation |
|--------------------|--------------------------------------|
| | $\langle \cos(t), \sin(t) \rangle$ |
| | $\langle \cos(t), t \rangle$ |
| | $\langle \cos(t), \cos^2(t) \rangle$ |
| | $\langle \cos(t), \sin(2t) \rangle$ |



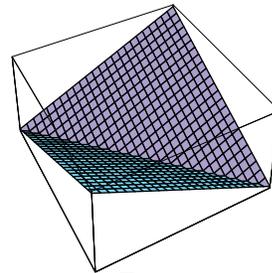
A



B



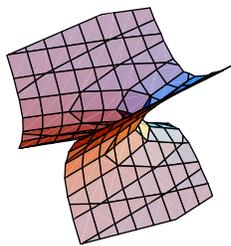
C



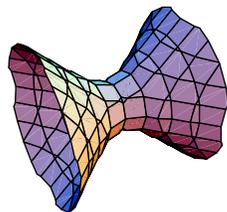
D

| Enter A,B,C,D here | Equation |
|--------------------|-----------------------|
| | $f(x, y) = x^2 - y^2$ |
| | $f(x, y) = x + y $ |
| | $f(x, y) = x^2 + y^2$ |
| | $f(x, y) = xy $ |

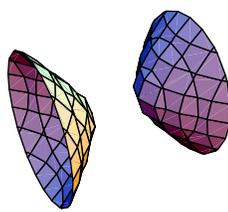
Solution:



I



II

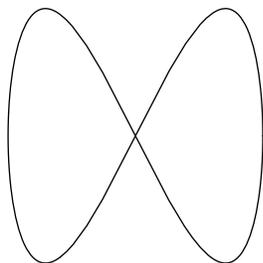


III

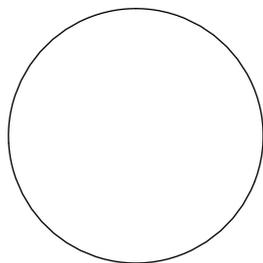


IV

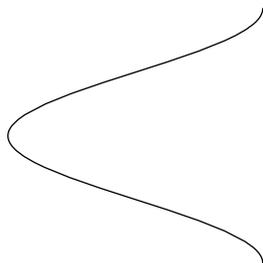
| Enter I,II,III,IV here | Equation |
|------------------------|----------------------------|
| I | $x + y^2 - z^2 - 1 = 0$ |
| II | $-x^2 + y^2 + z^2 - 1 = 0$ |
| III | $-x^2 + y^2 + z^2 + 1 = 0$ |
| IV | $-x + y^2 + z^2 + 1 = 0$ |



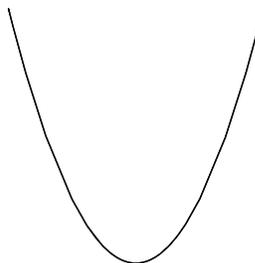
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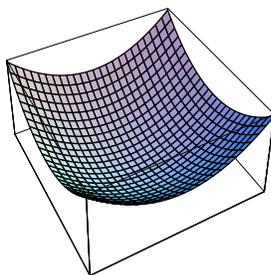


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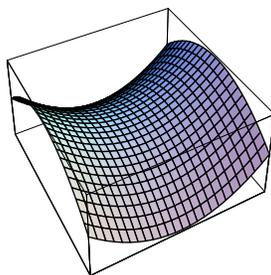


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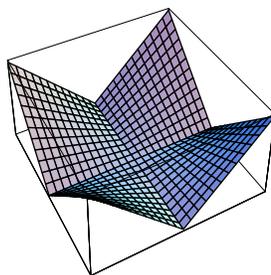
| Enter 1,2,3,4 here | Equation |
|--------------------|--------------------------------------|
| 2 | $\langle \cos(t), \sin(t) \rangle$ |
| 3 | $\langle \cos(t), t \rangle$ |
| 4 | $\langle \cos(t), \cos^2(t) \rangle$ |
| 1 | $\langle \cos(t), \sin(2t) \rangle$ |



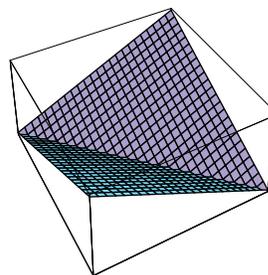
A



B



C



D

| Enter A,B,C,D here | Equation |
|--------------------|-----------------------|
| B | $f(x, y) = x^2 - y^2$ |
| D | $f(x, y) = x + y $ |
| A | $f(x, y) = x^2 + y^2$ |
| C | $f(x, y) = xy $ |

Problem 3) (10 points)

Imagine the planet Earth as the unit sphere in 3D space centered at the origin. An asteroid is approaching from the point $P = (0, 4, 3)$ along the path

$$\vec{r}(t) = \langle (4-t)\sin(t), (4-t)\cos(t), 3-t \rangle .$$

- a) When and where will it first hit the Earth?
 b) What velocity will it have at the impact?



Solution:

a) The distance to the origin $|\vec{r}(t)| = \sqrt{(4-t)^2 + (3-t)^2} = \sqrt{25 + 2t^2 - 14t}$ is equal 1 for $t = 3$ or $t = 4$.

b) The velocity is $\vec{r}'(t) = \langle (4-t)\cos(t) - \sin(t), -\cos(t) - (4-t)\sin(t), -1 \rangle$. The velocity at time $t = 3$ is $\langle \cos(3) - \sin(3), -\sin(3) - \cos(3), -1 \rangle$. (The speed at time $t = 3$ is $\sqrt{3}$.)

Problem 4) (10 points)

Find the distance between the z -axis and the line

$$L : \frac{x+2}{4} = \frac{y-1}{3} = \frac{z}{2} .$$

Solution:

The z axes can be parametrized as

$$\vec{r}(t) = P + t\vec{v} = \langle 0, 0, 0 \rangle + t\langle 0, 0, 1 \rangle$$

The line L can be parametrized as

$$\vec{r}(t) = Q + t\vec{w} = \langle -2, 1, 0 \rangle + t\langle 4, 3, 2 \rangle$$

The distance is the length of the projection of $\vec{PQ} = \langle -2, 1, 0 \rangle$ onto the normal vector $\vec{n} = \vec{v} \times \vec{w} = \langle -3, 4, 0 \rangle$. This is

$$d = \frac{|\langle -2, 1, 0 \rangle \cdot \langle -3, 4, 0 \rangle|}{|\langle -3, 4, 0 \rangle|} = 10/5 = 2 .$$

The answer is 2.

Problem 5) (10 points)

a) Find a parametrization $\vec{r}(t)$ of the line which is the intersection of the two planes

$$4x + 6y - z = 1$$

and

$$4x + z = 0 .$$

b) Find the point on the line which is closest to the origin.

Hint. What relation between $\vec{r}(t)$ and \vec{v} (a vector in the line) assures that \vec{r} is closest to the origin?

Solution:

a) In order to find the line of intersection, we have to find a point Q in the intersection as well as the direction of intersection. We get a point in the intersection by setting one variable zero. Lets take $x = 0$. Then $6y - z = 1, z = 0$ so that $Q = (0, 1/6, 0)$.

The cross product of the normal vectors between two vectors is perpendicular to the normal vectors of the plane. The vector $\vec{v} = \langle 4, 6, -1 \rangle$ is perpendicular to the plane $4x + 6y - z = 1$. The vector $\vec{w} = \langle 4, 0, 1 \rangle$ is perpendicular to the plane $4x + z = 0$. The vector $\vec{u} = \vec{v} \times \vec{w} = \langle 6, -8, -24 \rangle$ is a vector in the direction of the line. b) According to the hint, $r(t)$ and \vec{v} must be perpendicular, that is the dot product between $\vec{r}(t) = \langle 6t, 1/6 - 8t, -24t \rangle$ and $\vec{v} = \langle 4, 6, 1 \rangle$ is zero. This gives $t = 1/507$. The closest point is $\vec{r}(t) = (0, 1/6, 0) + (6, -8, -24)/507$.

Problem 6) (10 points)

Consider the parameterized curve

$$\vec{r}(t) = \langle e^t + e^{-t}, 2 \cos(t), 2 \sin(t) \rangle .$$

Find the arc length of this curve from $t = 0$ to $t = 4$.

Solution:

The velocity is $\vec{r}'(t) = \langle e^t - e^{-t}, -2 \sin(t), 2 \cos(t) \rangle$. The speed is $\sqrt{2 + e^{-2t} + e^{2t}} = (e^t + e^{-t})$. The integral

$$L = \int_0^4 (e^t + e^{-t}) dt$$

gives $e^4 - e^{-4} = 2 \sinh(4)$.

Problem 7) (10 points)

The set of points P for which the distance from P to $A = (1, 2, 3)$ is equal to the distance from P to $B = (5, 8, 5)$ forms a plane S .

- a) Find the equation $ax + by + cz = d$ of the plane S .
- b) Find the distance from A to S .

Solution:

a) The key insight is that the point $Q = (A + B)/2 = (3, 5, 4)$ is in the middle of the two points. The plane has to pass through this point. The normal vector is parallel to $\vec{n} = \langle 4, 6, 2 \rangle$. The equation of the plane is

$$4x + 6y + 2z = 50 .$$

b) The distance from A to S is half the distance from A to B which is $|\vec{AB}|/2 = |\langle 4, 6, 2 \rangle|/2 = \sqrt{56}/2$.

| |
|------------------------|
| Problem 8) (10 points) |
|------------------------|

The Swiss tennis player Roger Federer hits the ball at the point $\vec{r}(0) = (0, 0, 3)$. The initial velocity is $\vec{r}'(0) = \langle 100, 10, 13 \rangle$. The tennis ball experiences a constant acceleration $\vec{r}''(t) = \langle 2, 0, -32 \rangle$ which is due to the combined force of gravity and a constant wind in the x direction.

- a) Where does the tennis ball hit the ground $z = 0$?
- b) What is the z -component = (projection onto z vector) $P_{\vec{k}}(\vec{r}'(t))$ of the ball velocity at the impact?



Solution:

a) This is a typical free fall problem. After integrating twice the equation $\vec{r}''(t) = \langle 2, 0, -32 \rangle$, we get

$$\vec{r}(t) = (0, 0, 3) + t(100, 10, 13) + t^2(1, 0, -16)$$

$$\vec{r}'(t) = (100, 10, 13) + 2t(1, 0, -16)$$

We get an impact with the ground $z = 0$ at time $t = 1$. This is at the position $\vec{r}(1) = (101, 10, 0)$.

b) The velocity at time $t = 1$ is $(102, 10, -19)$. The projection onto the vector \vec{k} is $(0, 0, -19)$. Note that this is a vector. The z -component of this vector is the third component of this vector which is -19 .

Problem 9) (10 points)

a) (6 points) Parameterize the intersection of the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 2$$

with the plane $z = 3$.

b) (4 points) What is the curvature of the curve at the point $(2, 0, 3)$?

Hint. While you can use the curvature formula $\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$ you are also allowed to cite a fact which you know about the curvature.

Solution:

a) The parametrization is $\vec{r}(t) = \langle 2 \cos(t), 2 \sin(t), 3 \rangle$. This is a circle of radius 2.

b) The curvature is $1/2$ at all points.