Math21a Review

for first hourly

October 15, 2006

Oliver Knill

Multivariable calculus!

I wanted first to theme this review on a recent movie:

number two

which contains a lot of Multivariable calculus:

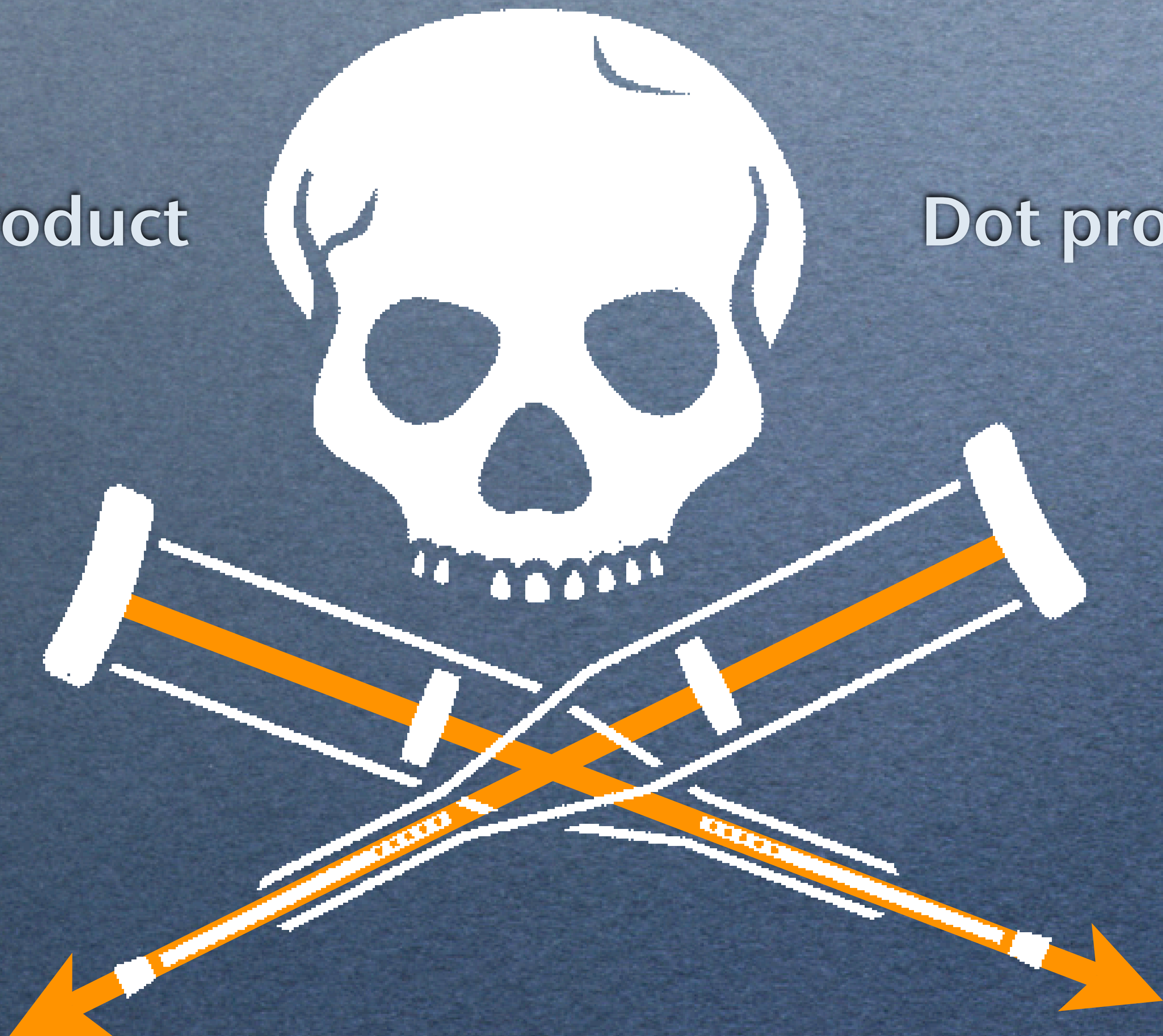
but often streches
the limit of good taste:

So, after visiting the cirque soleil yesterday, here is an other theme.

First some geometry

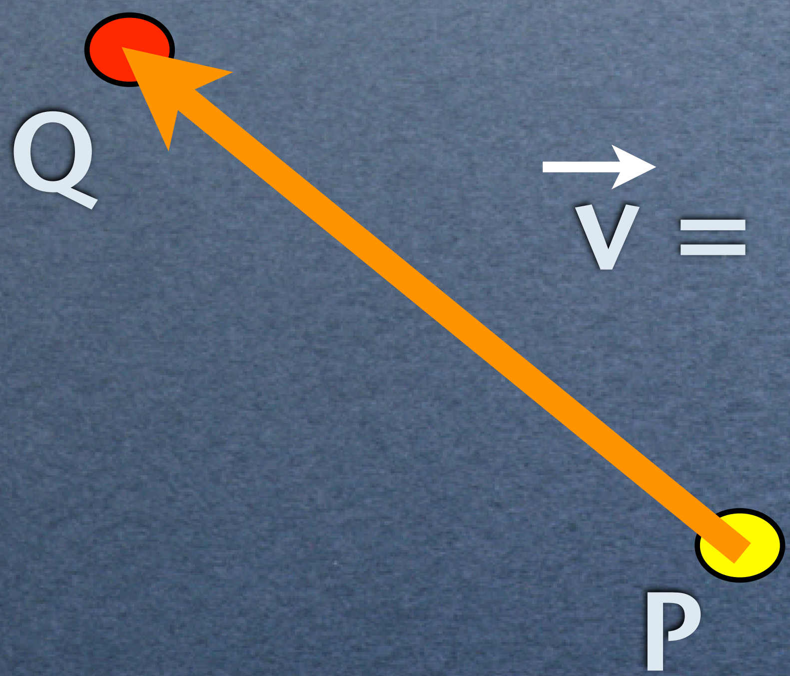
Cross product

Dot product



Points - Vectors

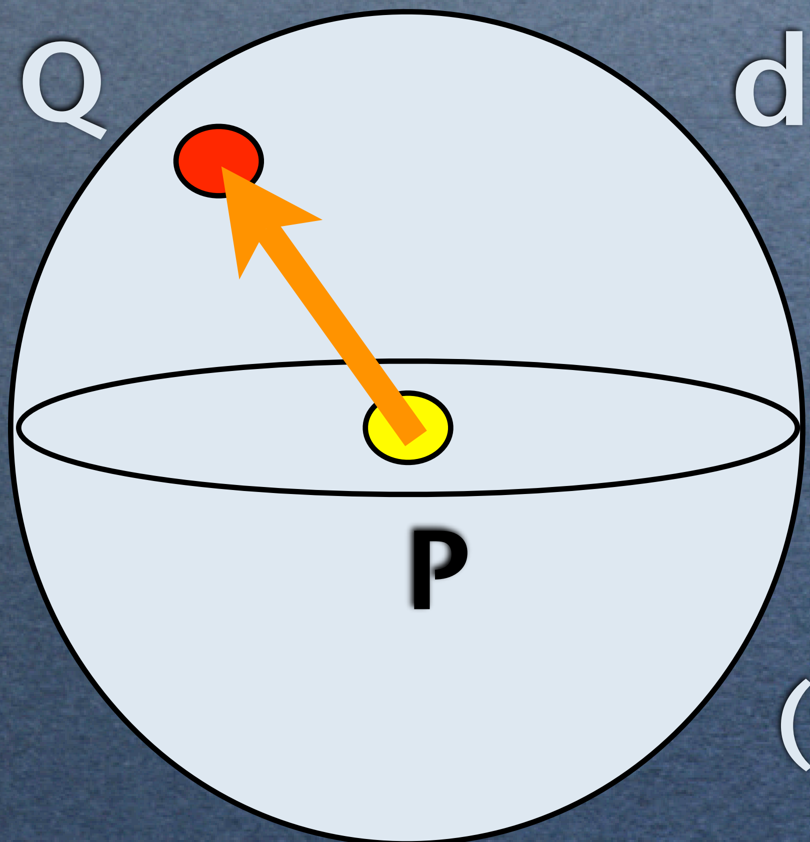
P = (3,4 6), Q =(5,4,7)



$\vec{v} = \langle 2,0,1 \rangle = \overrightarrow{PQ}$

Distances-Spheres

P = (3,4 6), Q =(5,4,7)



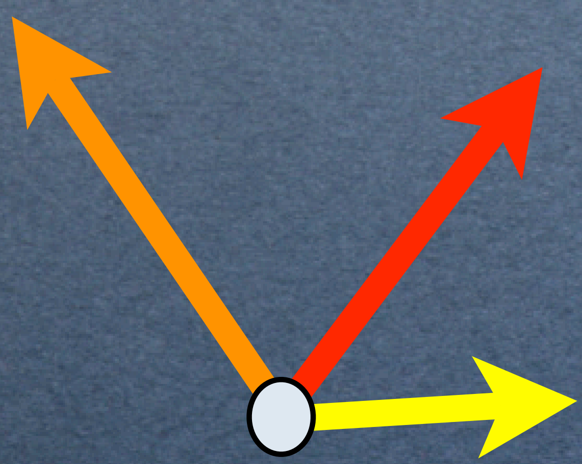
$d(P,Q) = |\overrightarrow{PQ}|$

$= (4+0+1)^{1/2}$

$(x-3)^2 + (y-4)^2 + (z-6)^2 = 5$

Dot -Cross Products

$\vec{v} = < 3,4\ 6>, \vec{w} = <5,4,7>$



$\vec{v} \cdot \vec{w} = 73$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 6 \\ 5 & 4 & 7 \end{vmatrix}$$

$= <4,9,-8>$

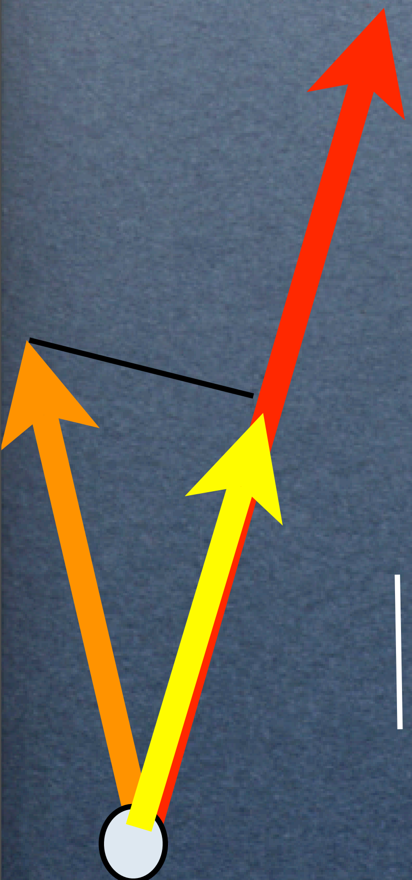
Formulas

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos(\alpha)$$

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin(\alpha)$$

Projection

$\vec{v} = < 1,2,3 >, \vec{w} = <1,0-1>$



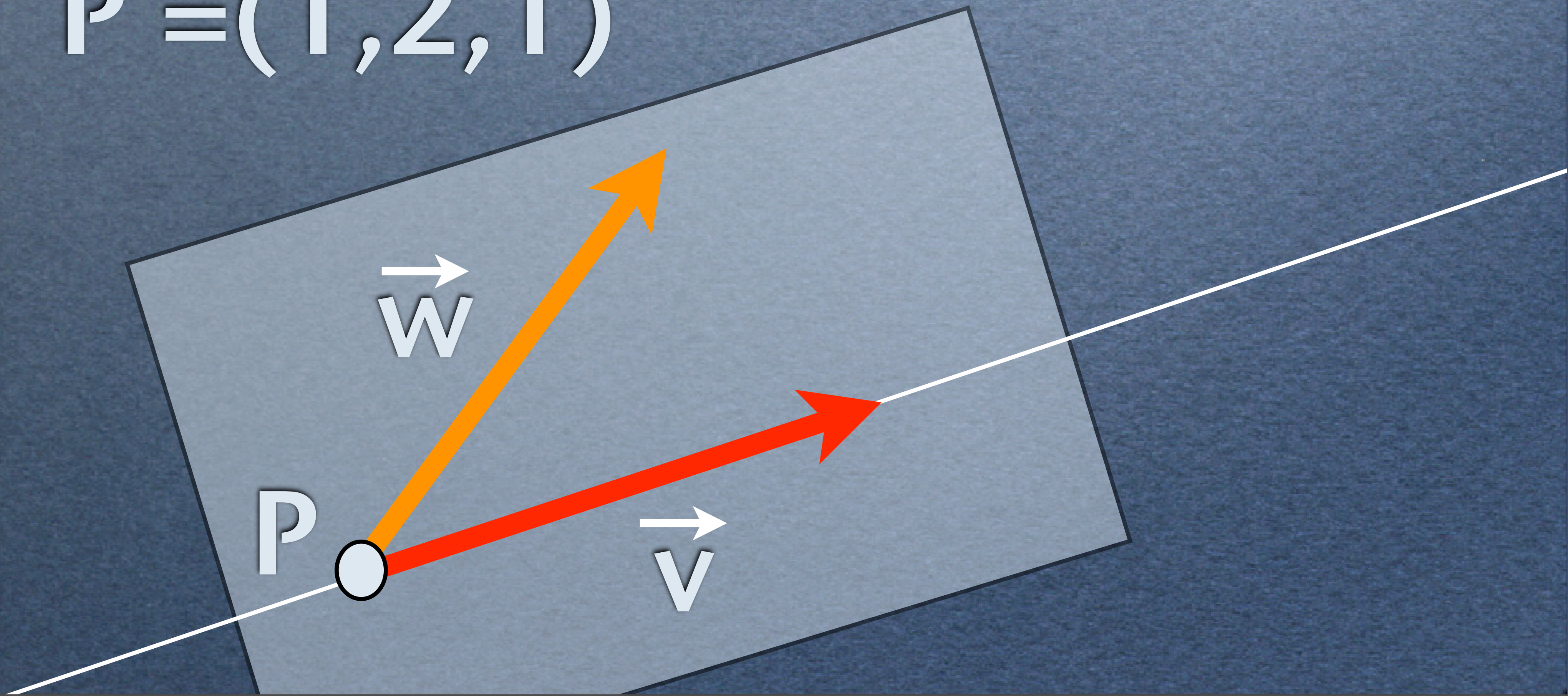
$$P_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w} = <-1,0,1>$$

$$|P_{\vec{w}}(\vec{v})| = \frac{|\vec{v} \cdot \vec{w}|}{|\vec{w}|} = \sqrt{2}$$

Lines and Planes

$\vec{v} = < 1,2,3 >, \vec{w} = <1,0-1>$

$P = (1,2,1)$



Lines

$\vec{v} = < 1,2,3 >$ $P = (1,2,1)$

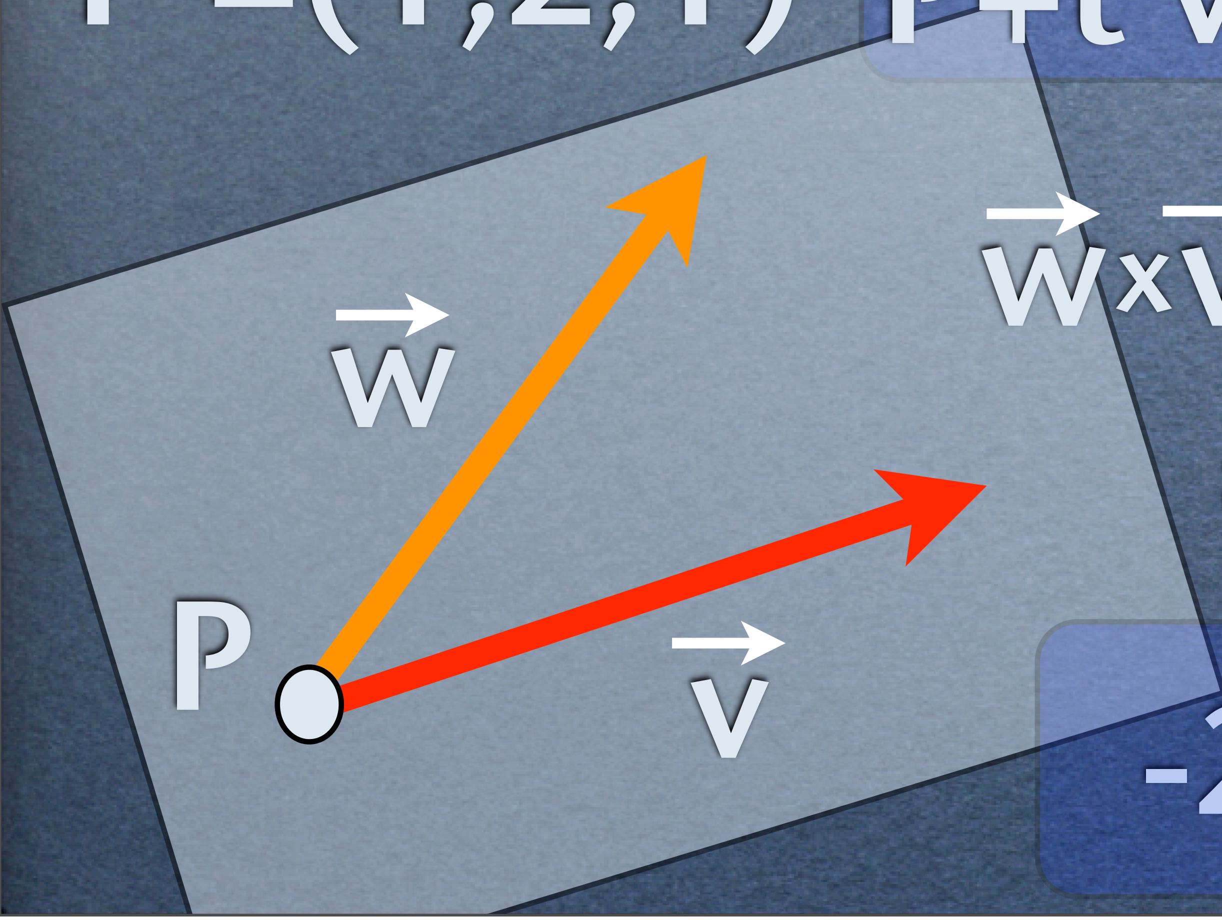
$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$$

$\vec{r}(t) = <1+t, 2+2t, 1+3t> = <x,y,z>$

Planes

$\vec{v} = \langle 1,2,3 \rangle, \ \vec{w} = \langle 1,0-1 \rangle$

$P = (1,2,1)$ $\quad P + t\,\vec{v} + s\,\vec{w} = \vec{r}(t,s)$



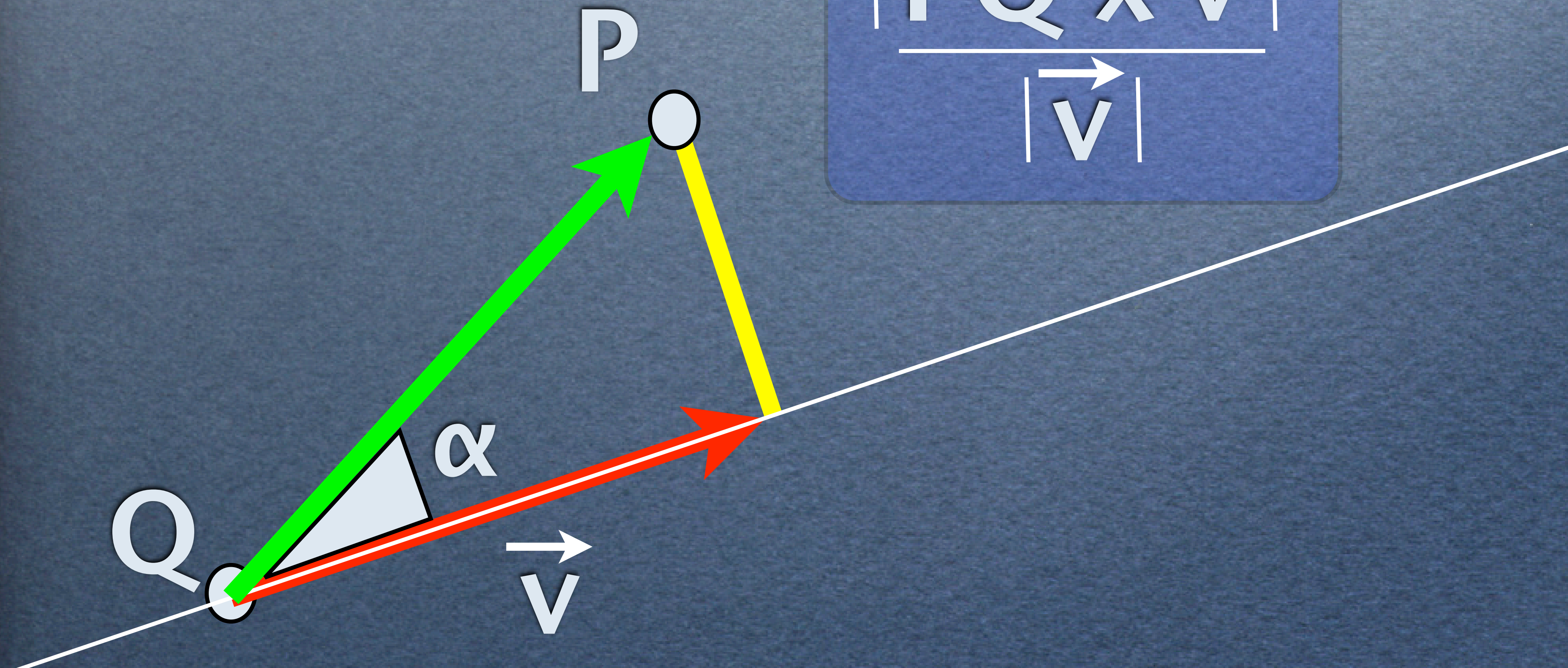
$\vec{w} \times \vec{v} = \langle a,b,c \rangle$

$= \langle -2,4,-2 \rangle$

$-2x+4y-2z=4$

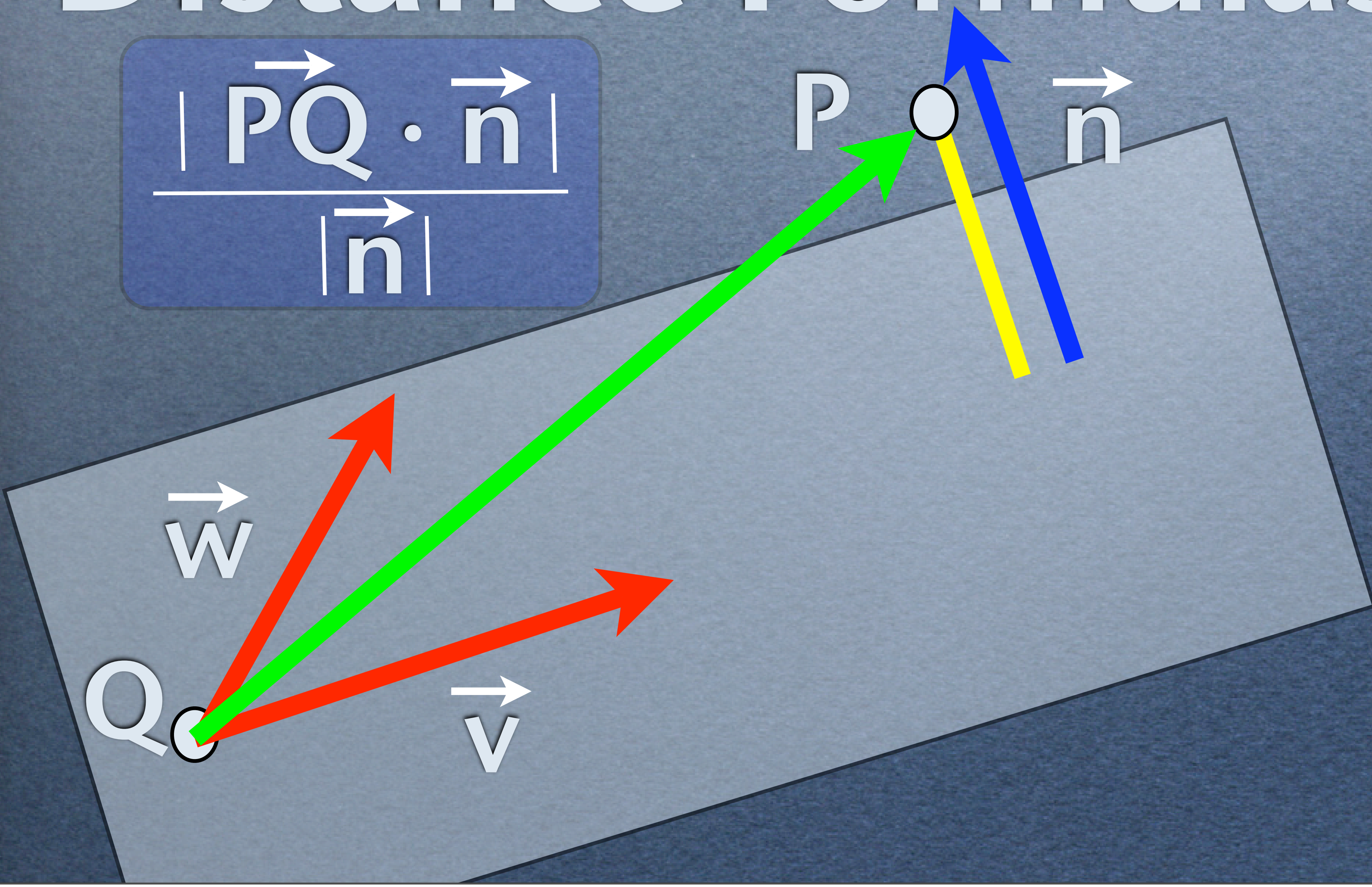
Distance Formulas

$$\frac{|\overrightarrow{PQ} \times \vec{v}|}{|\vec{v}|}$$

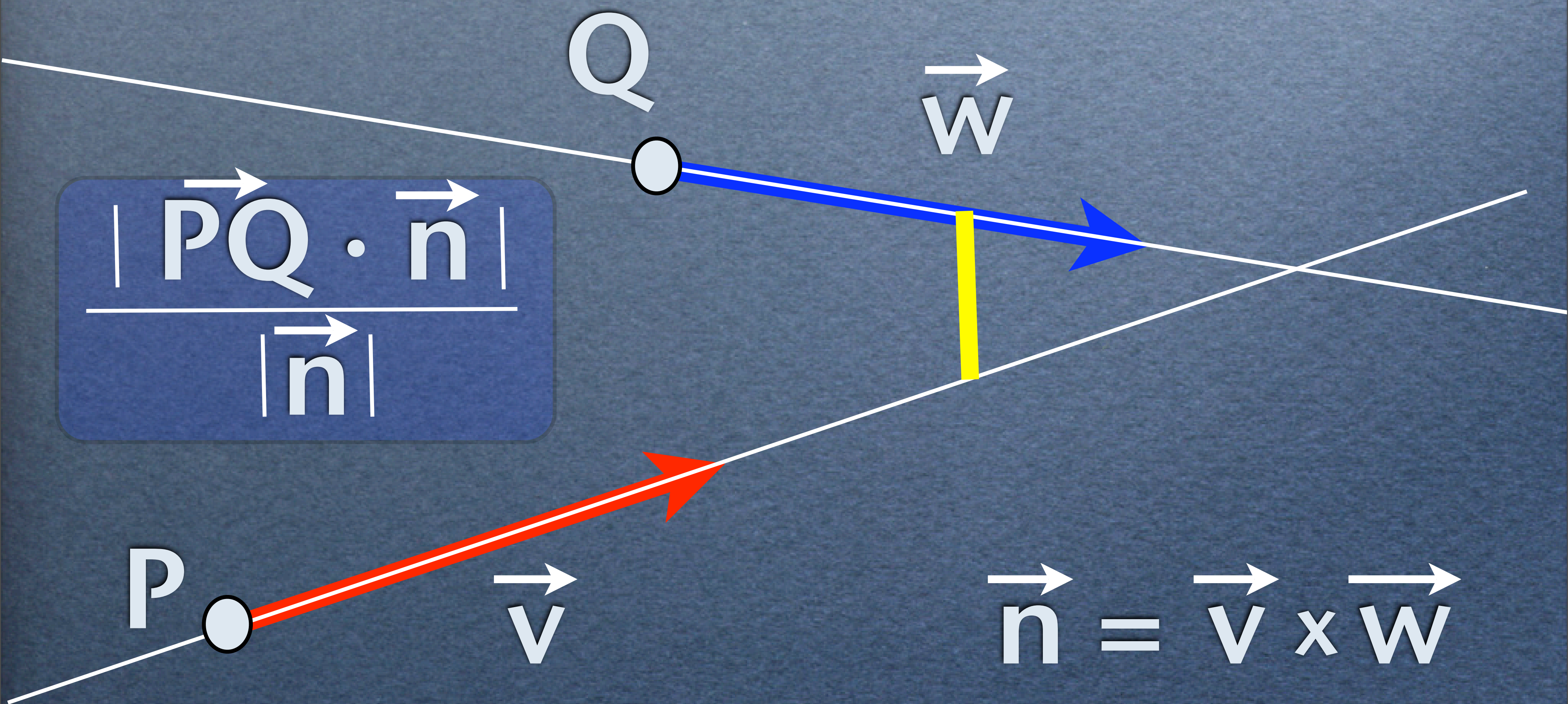


Distance Formulas

$$\frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|}$$



Distance Formulas



$$\frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

$$\vec{n} = \vec{v} \times \vec{w}$$

Problem 1

Done on blackboard

Given the points

A=(2,3,4),B=(3,4,5) ,C=(1,2,1), D=(3,2,1)

Find the distance between the line through A,B and the line through C,D.

You might think now:

Since Knill used this example, it will not appear in the test. However, remember that I know this too. To make a surprise, we could have put this example in the test. But because I mentioned this now, it most likely will not. But then again, it would be a surprise if it would appear...

By the way:

Harvard does not do “Surprise exams”, because surprise exams promised on a fixed time interval do not exist:

Proof. Assume we would announce now that there will be a surprise exam between now and October 31 and that it has to be a surprise.

October 31 can not be an exam date:

Proof. If it were on 31 October then in the night of 30-31 October, all of you would know that the exam would be the next day. It would not be a surprise exam.

October 30 is out:

Proof. If it were on 30 October then in the night of 29-30 October, all of you would know that the exam has to be the next day because we have already seen that it can not be on 31 March. It would not be a surprise exam.

etc. etc.


Clear?
Photo: Satoshi Kuribajashi

Area and Volume

$\vec{u} = < 1,2,3 >$, $\vec{v} = < 1,0-1 >$

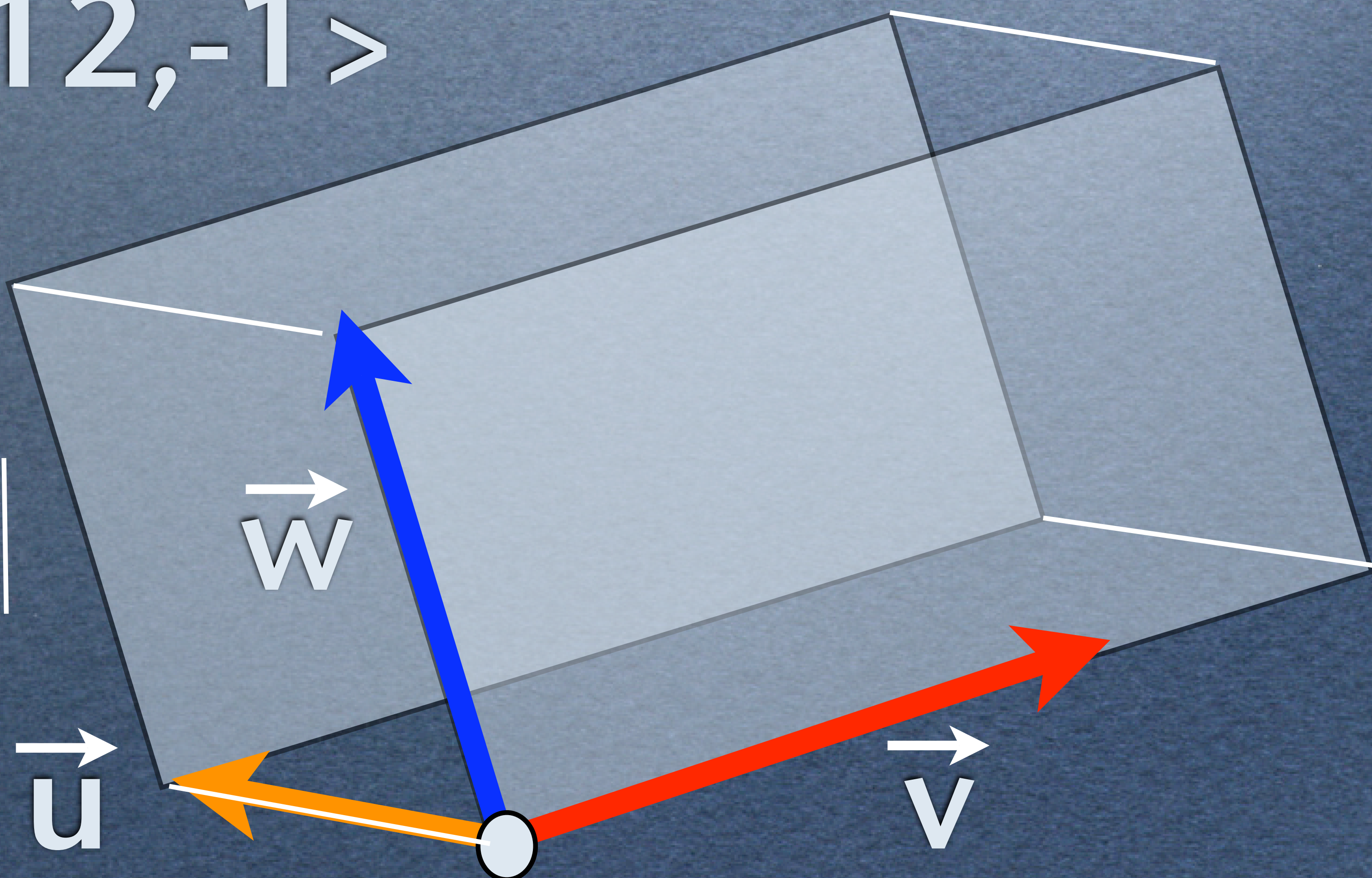
$\vec{w} = < -12,-1 >$

$|\vec{u} \times \vec{v}|$

area

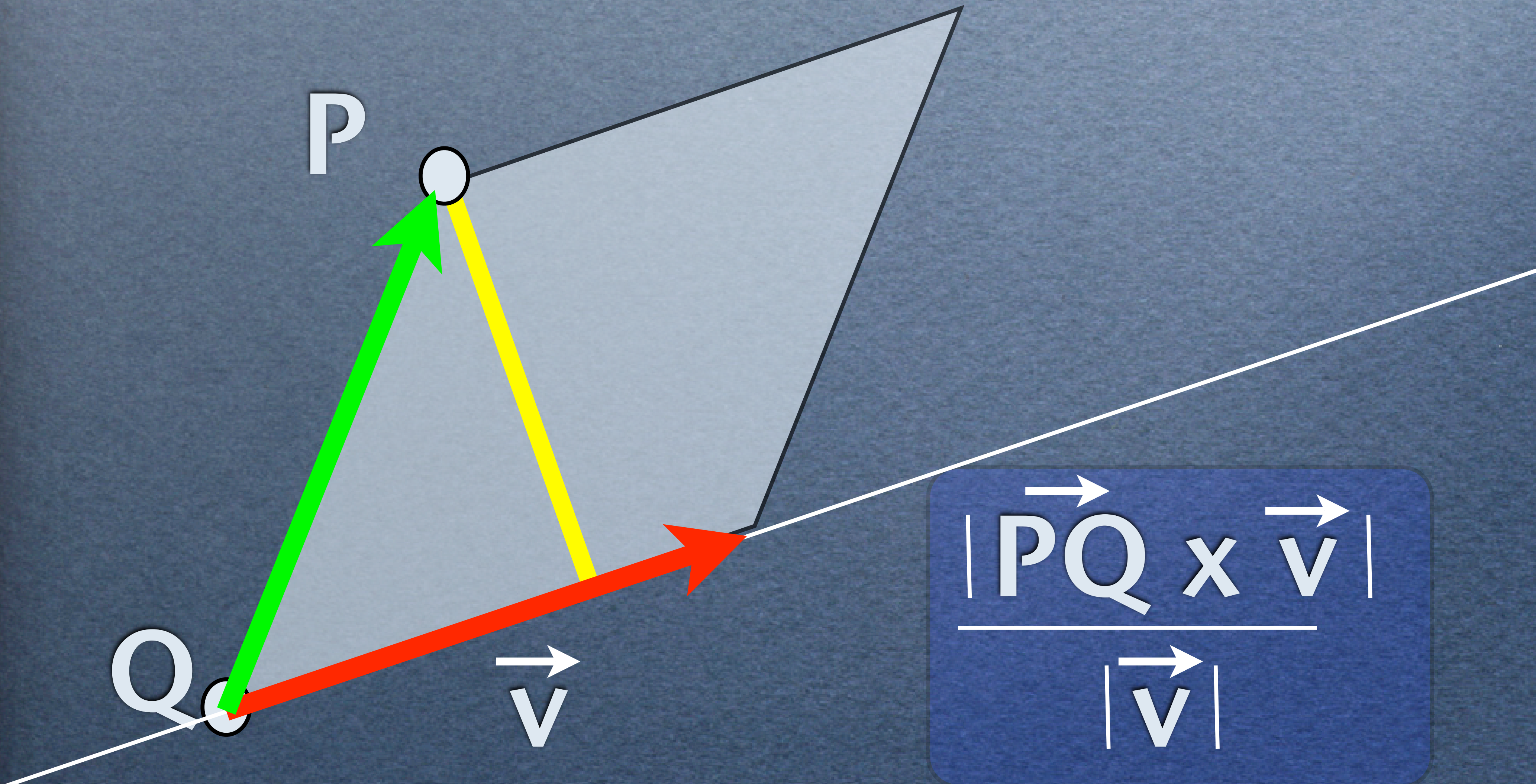
$|\vec{w} \cdot \vec{u} \times \vec{v}|$

volume



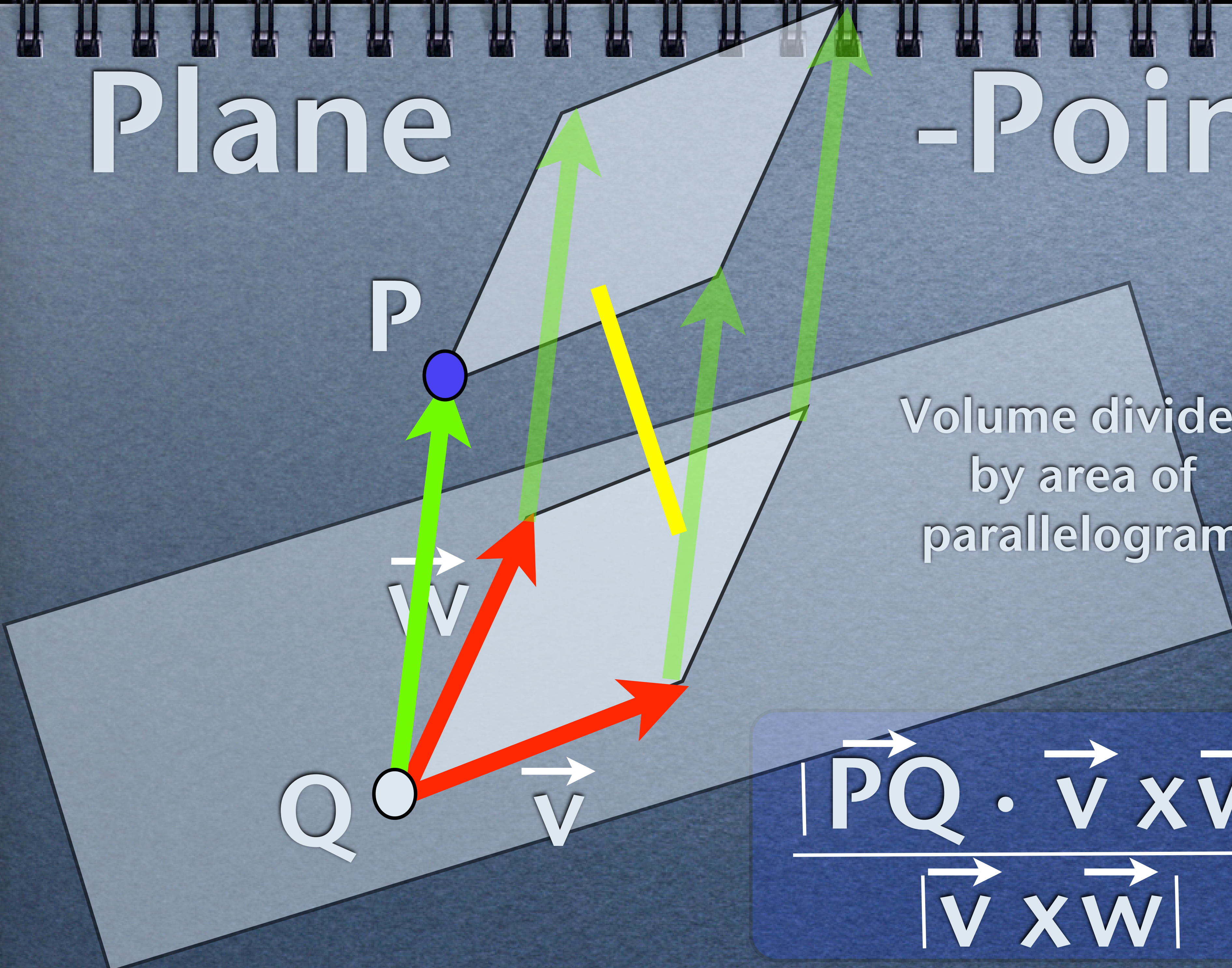
Distance formulas derived from area and volume:

Distance Line-Point



$$\frac{|\overrightarrow{PQ} \times \vec{v}|}{|\vec{v}|}$$

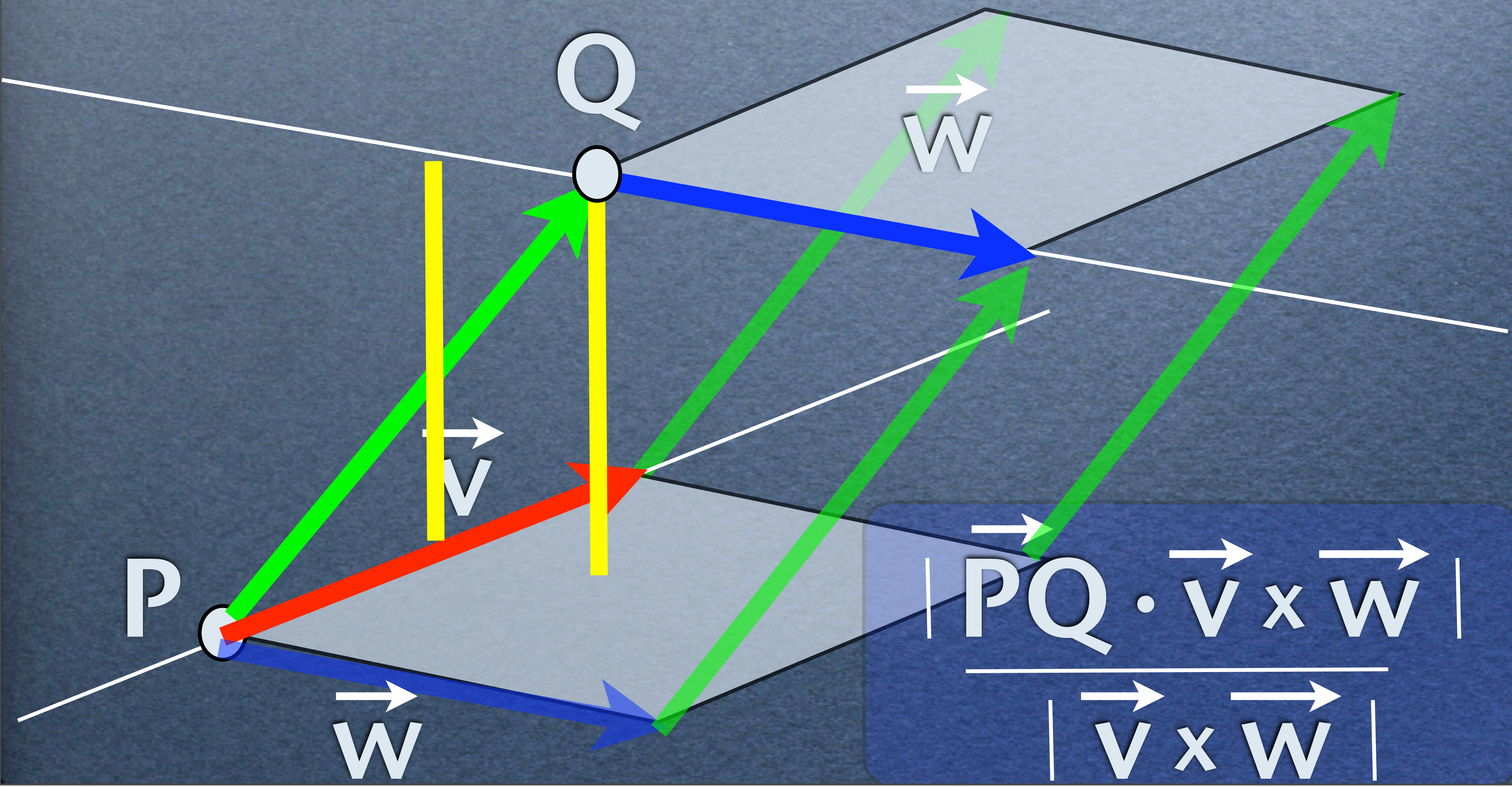
Plane -Point



Volume divided by area of parallelogram

$$\frac{|\vec{PQ} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

Line - Line



$$\frac{|\vec{PQ} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

Problem 2

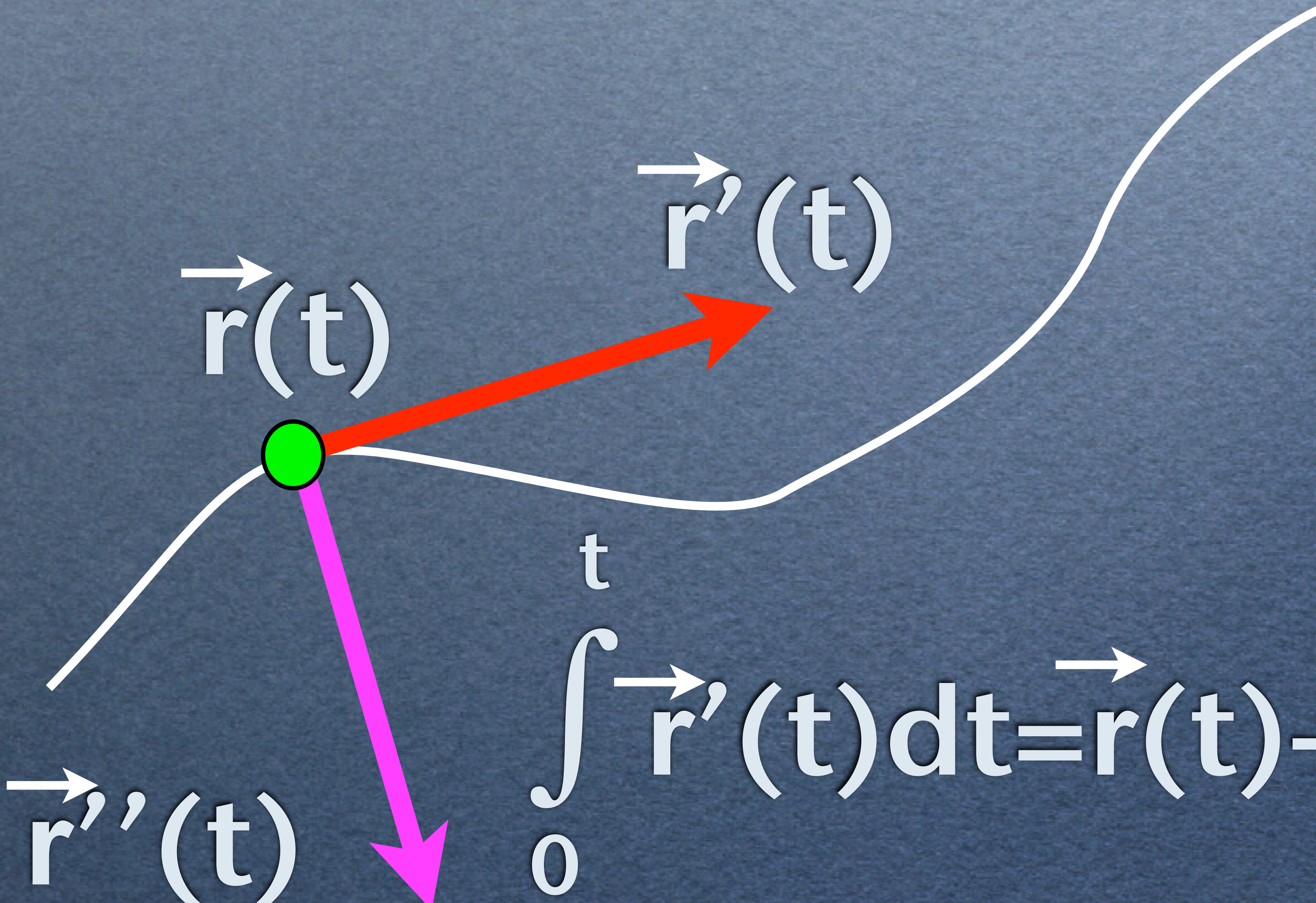
Done on blackboard

Find the line which is the reflection of

$\vec{r}(t) = (1,1,5) + t\,(3,4,5)$

at the xy-plane.

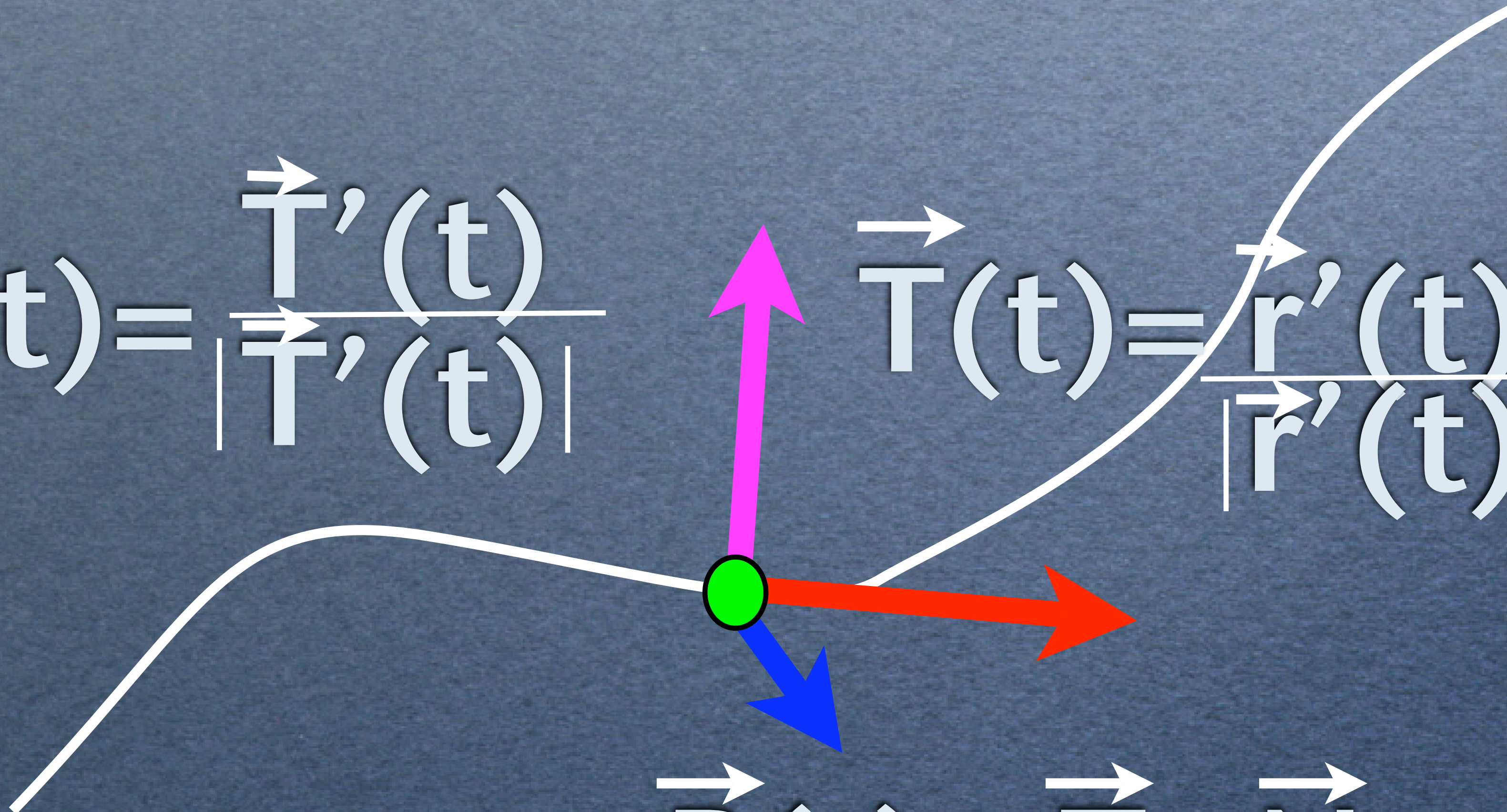
Velocity-Acceleration



$$\int_0^t \vec{r}'(t)\,dt = \vec{r}(t) - \vec{r}(0)$$

TNB- Frame

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

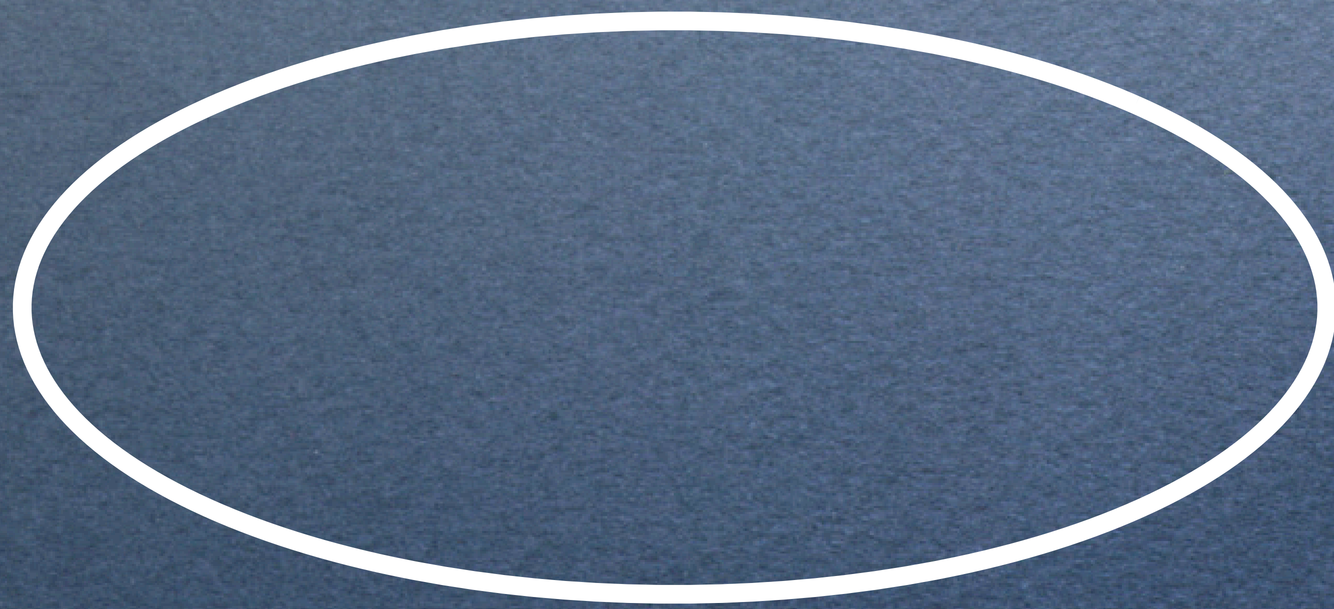


$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\vec{B}(t) = \vec{T} \times \vec{N}$$

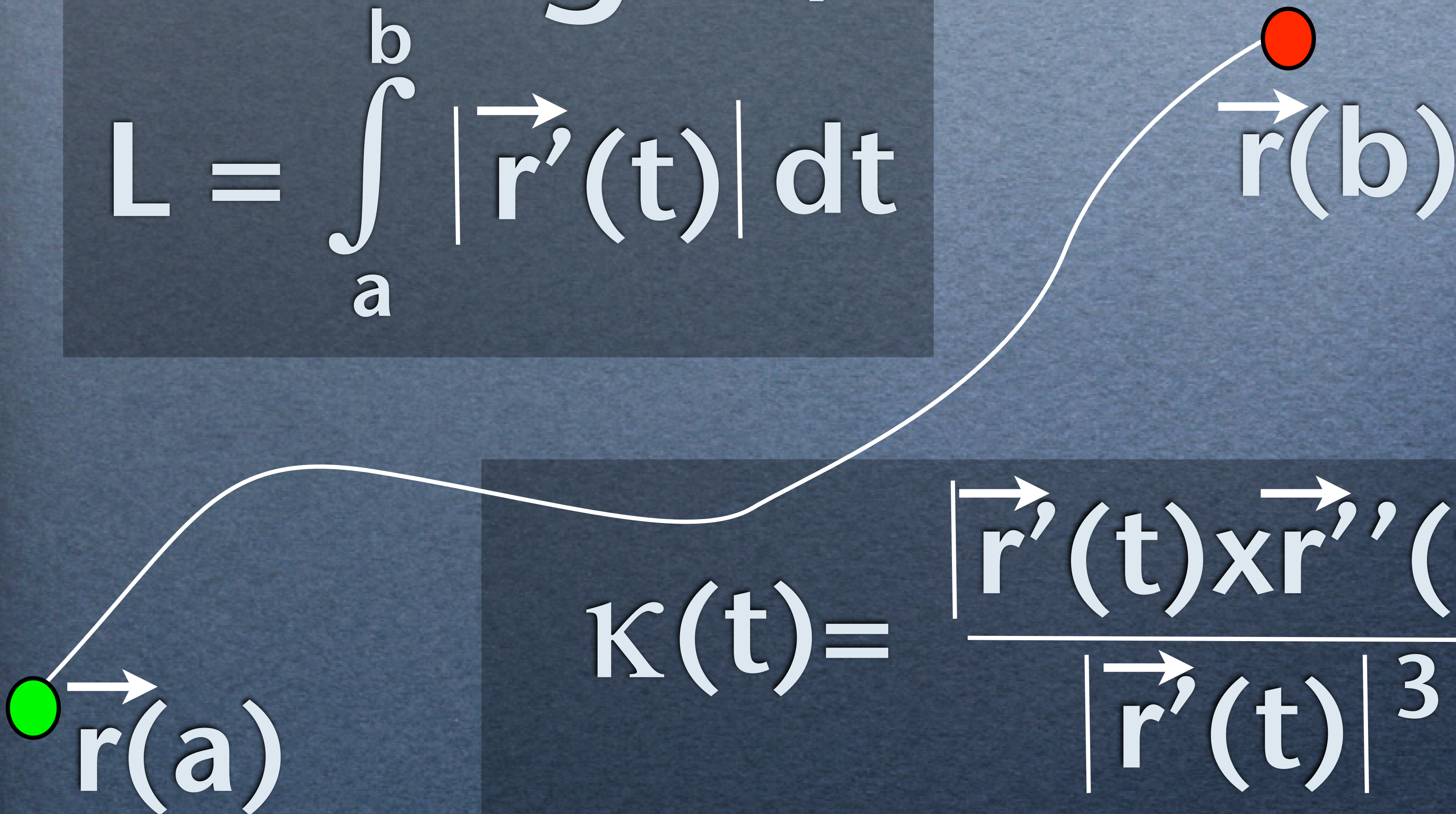
Important Example

$\vec{r}(t) = < a\cos(t), b\sin(t) >$



Arclength/curvature

$$L = \int_a^b \left|\vec{r}'(t)\right| dt$$



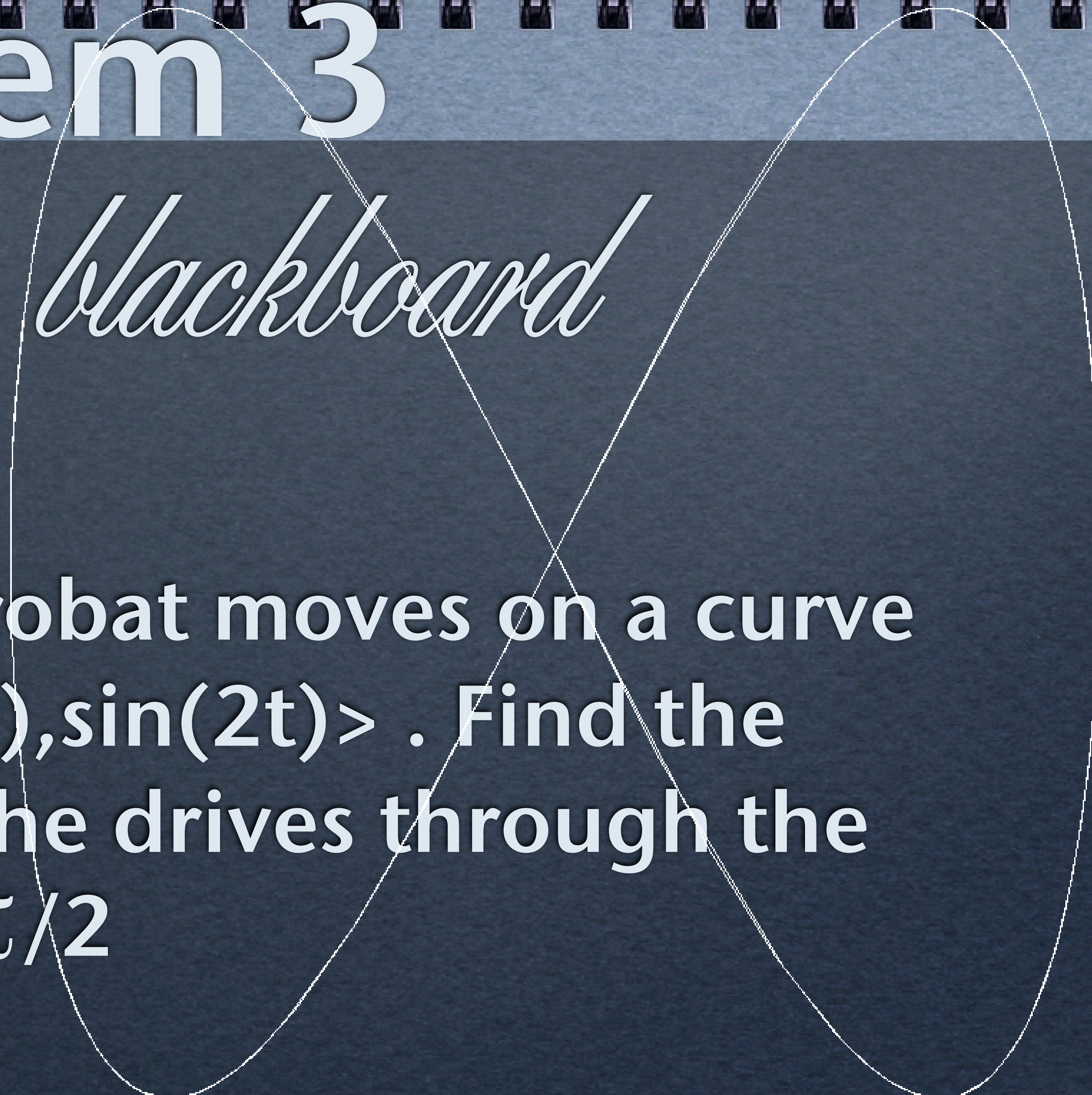
$$\kappa(t) = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3}$$

Independent of parametrization

Large or Small Curvature ?

Problem 3

Done on blackboard

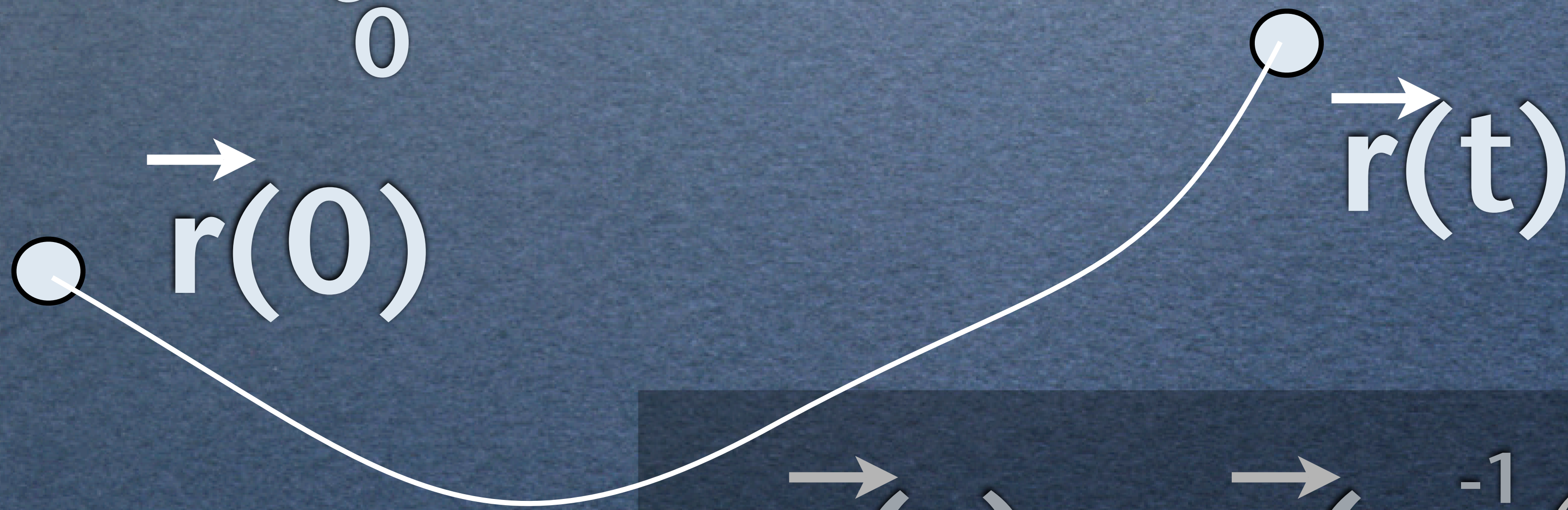


The wheel acrobat moves on a curve $\vec{r}(t) = \langle 2\cos(t), \sin(2t) \rangle$. Find the speed when she drives through the origin at $t = \pi/2$

Now to something really scary:

Arc Length Parametrization

$$\sigma(t) = \int_0^t \left|\vec{r}'(u)\right| du$$



$$\vec{p}(s) = \vec{r}(\sigma^{-1}(s))$$

Illustration:

Reparametrized

Problem 4

Done on blackboard

Find the arc length integral of the intersection of $x^2+4z^2= 4$ and $y-x\ z=8$

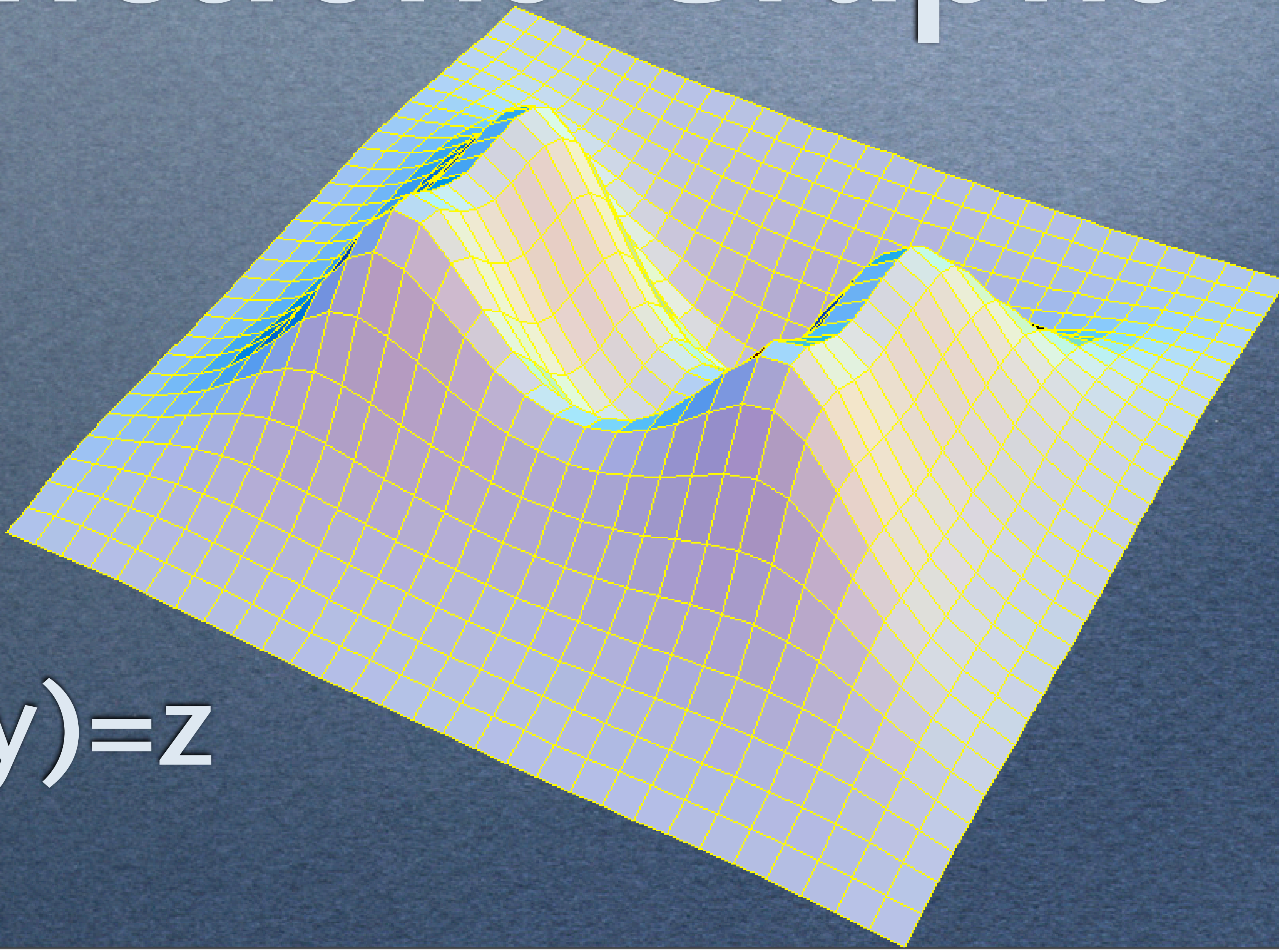
Free Fall

Problem 5

Done on blackboard

The little person on the balloon experiences a gravitational force of (0,0,-2) and is thrown with an initial velocity (2,3,4) starting
at (10,10,0). How high will it fly?

Functions-Graphs



$f(x,y)=z$

How to analyze ?

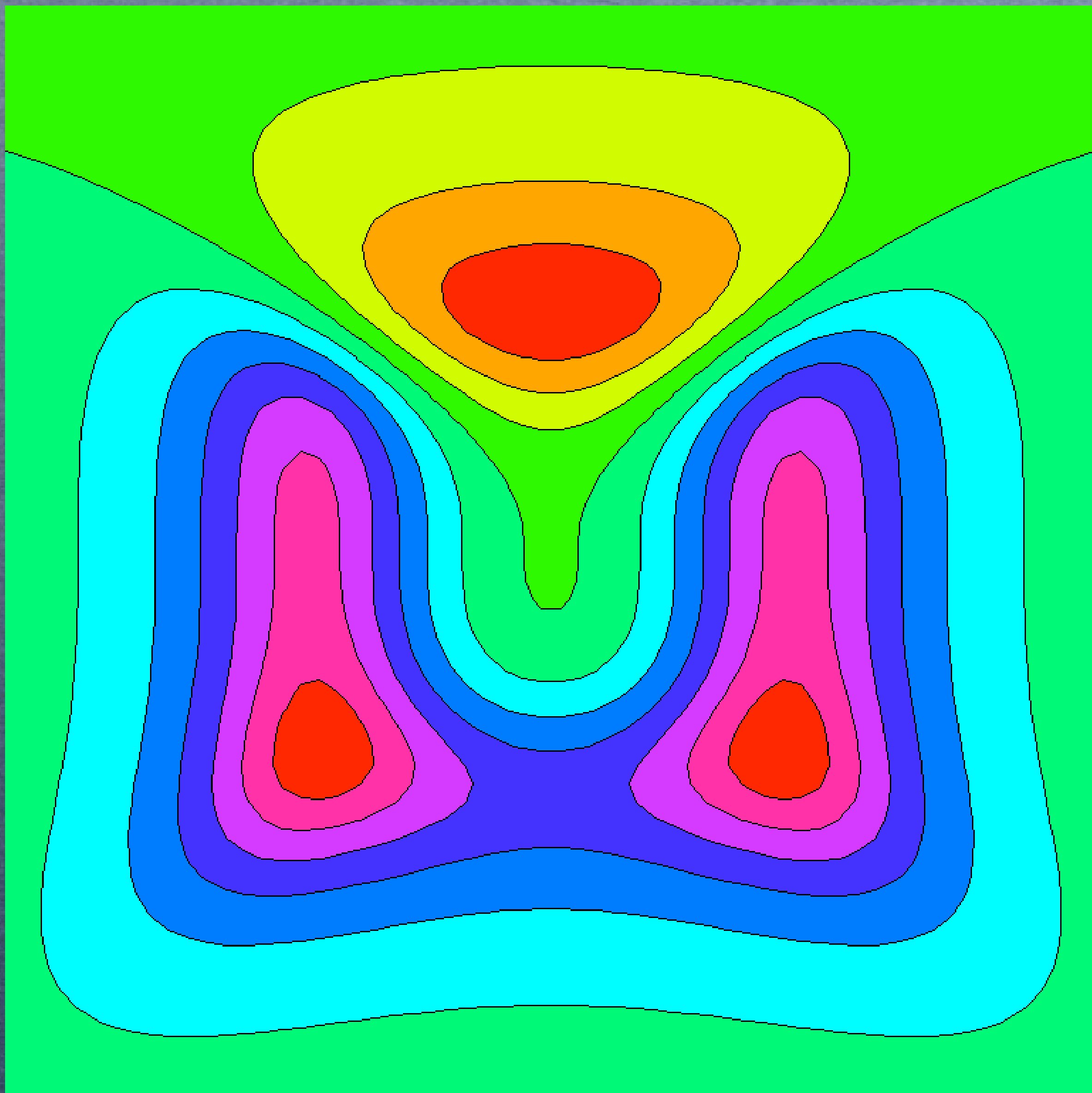
Traces

Contour lines

Symmetries

Asymptotic behavior

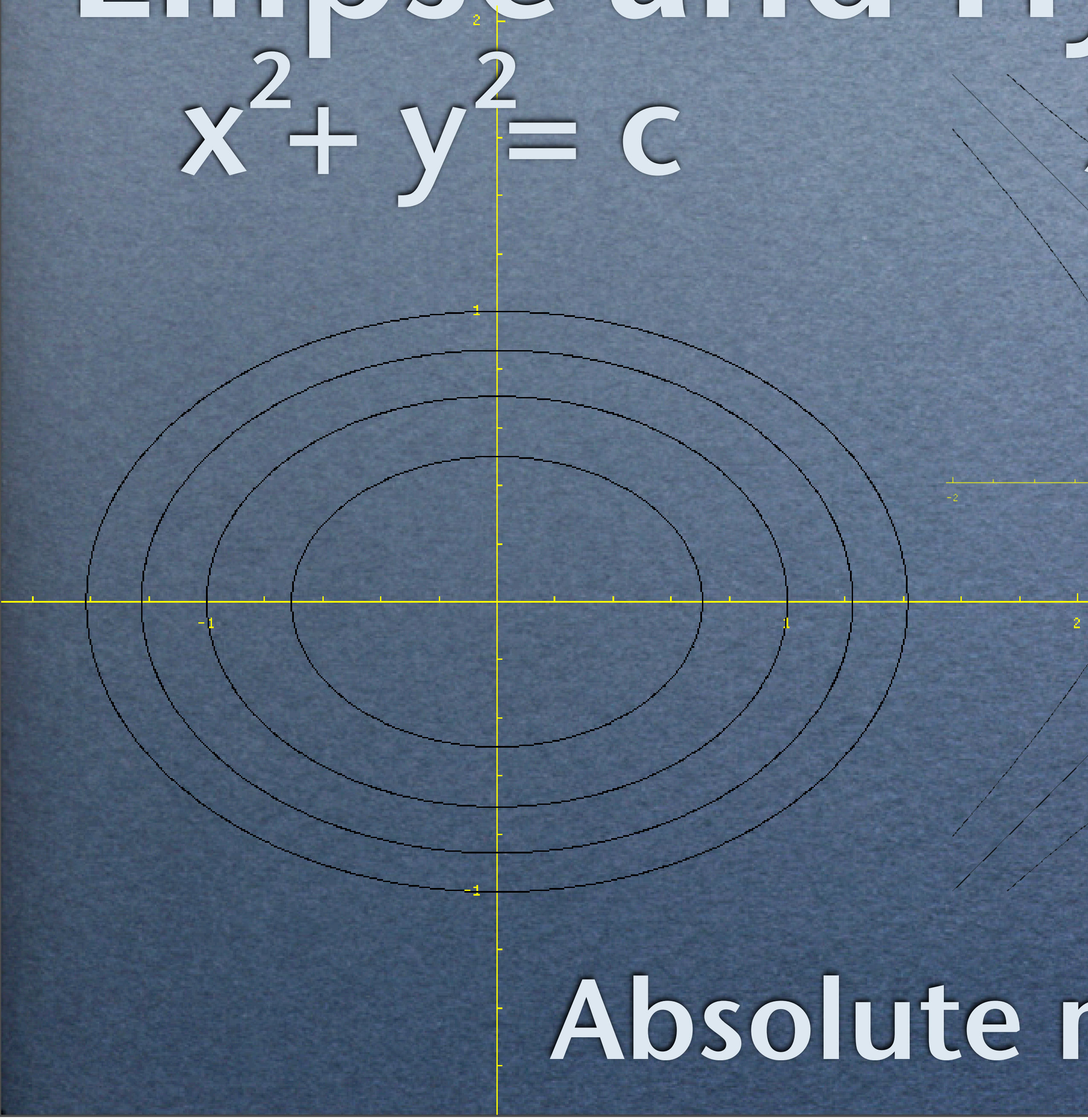
Contour Lines



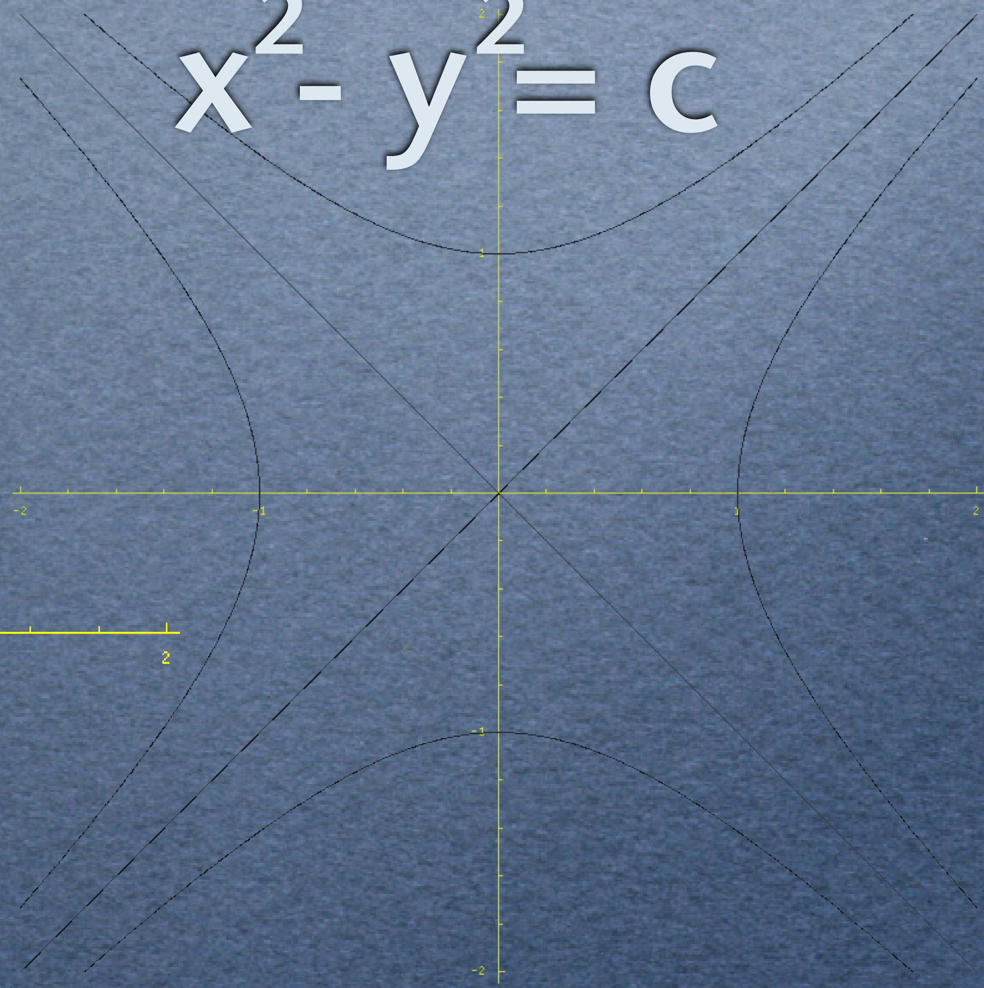
$f(x,y)=c$

Ellipse and Hyperbola

$x^2 + y^2 = c$



$x^2 - y^2 = c$

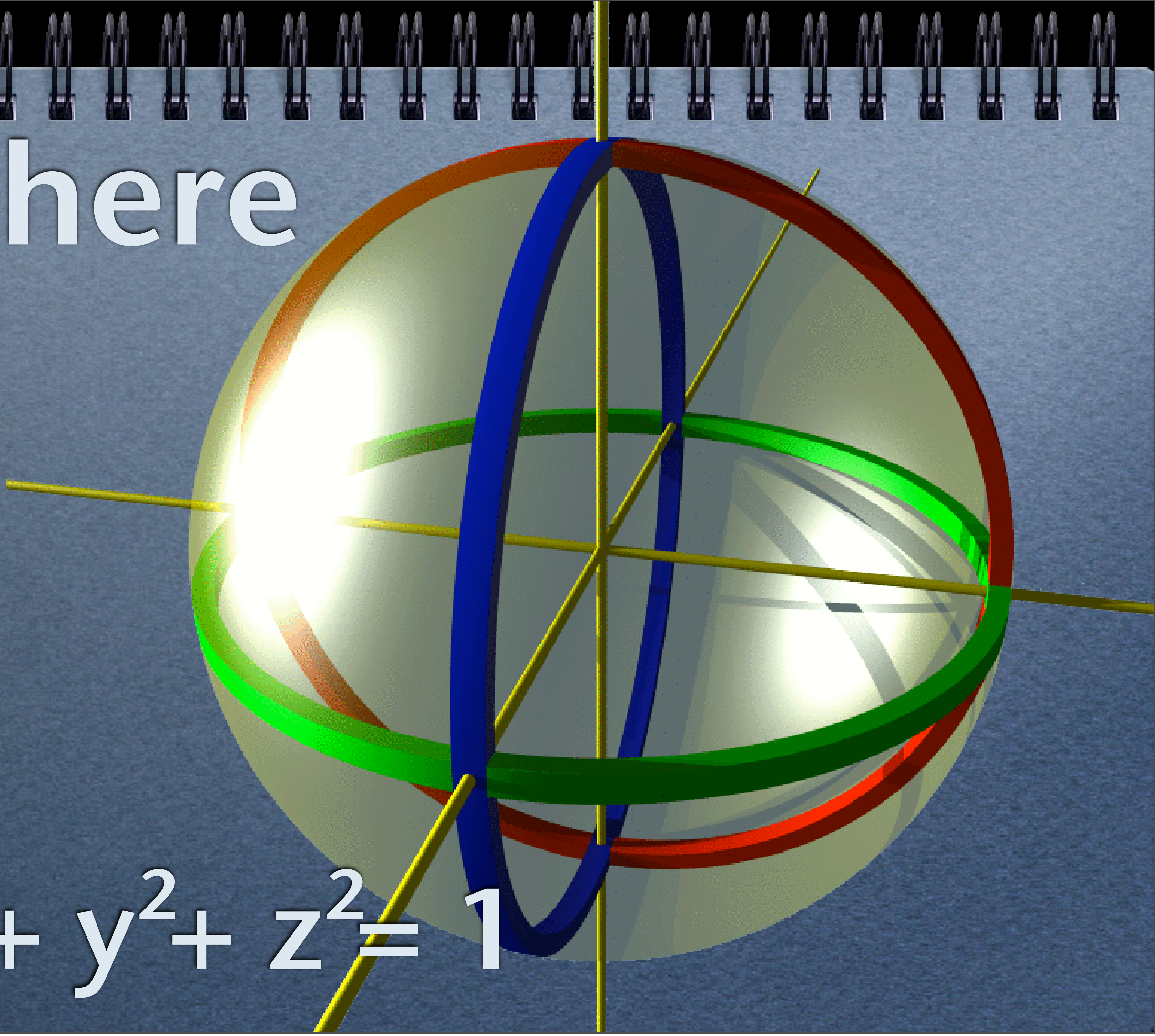


Absolute must !

Contour Surfaces

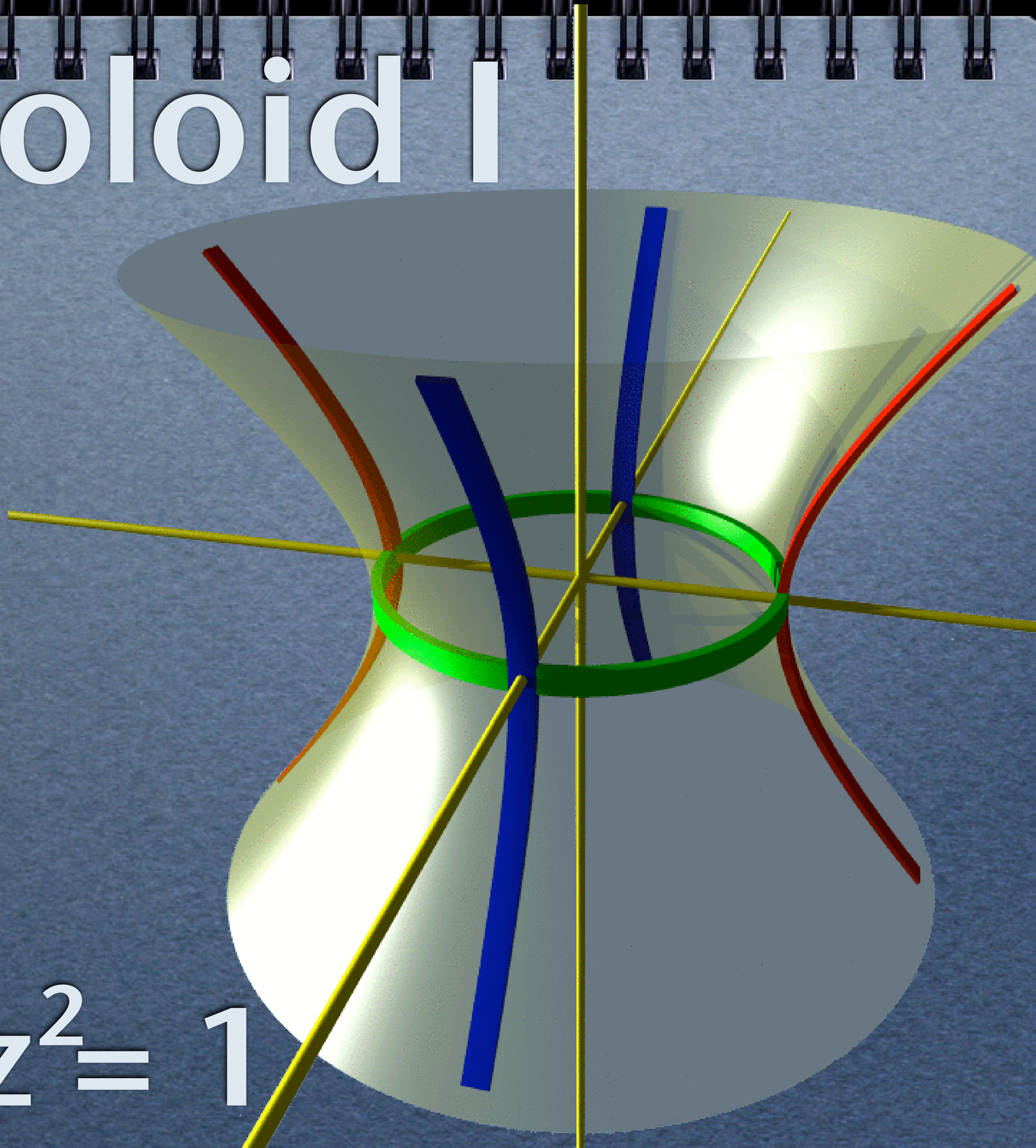
$$f(x,y,z)=c$$

Sphere



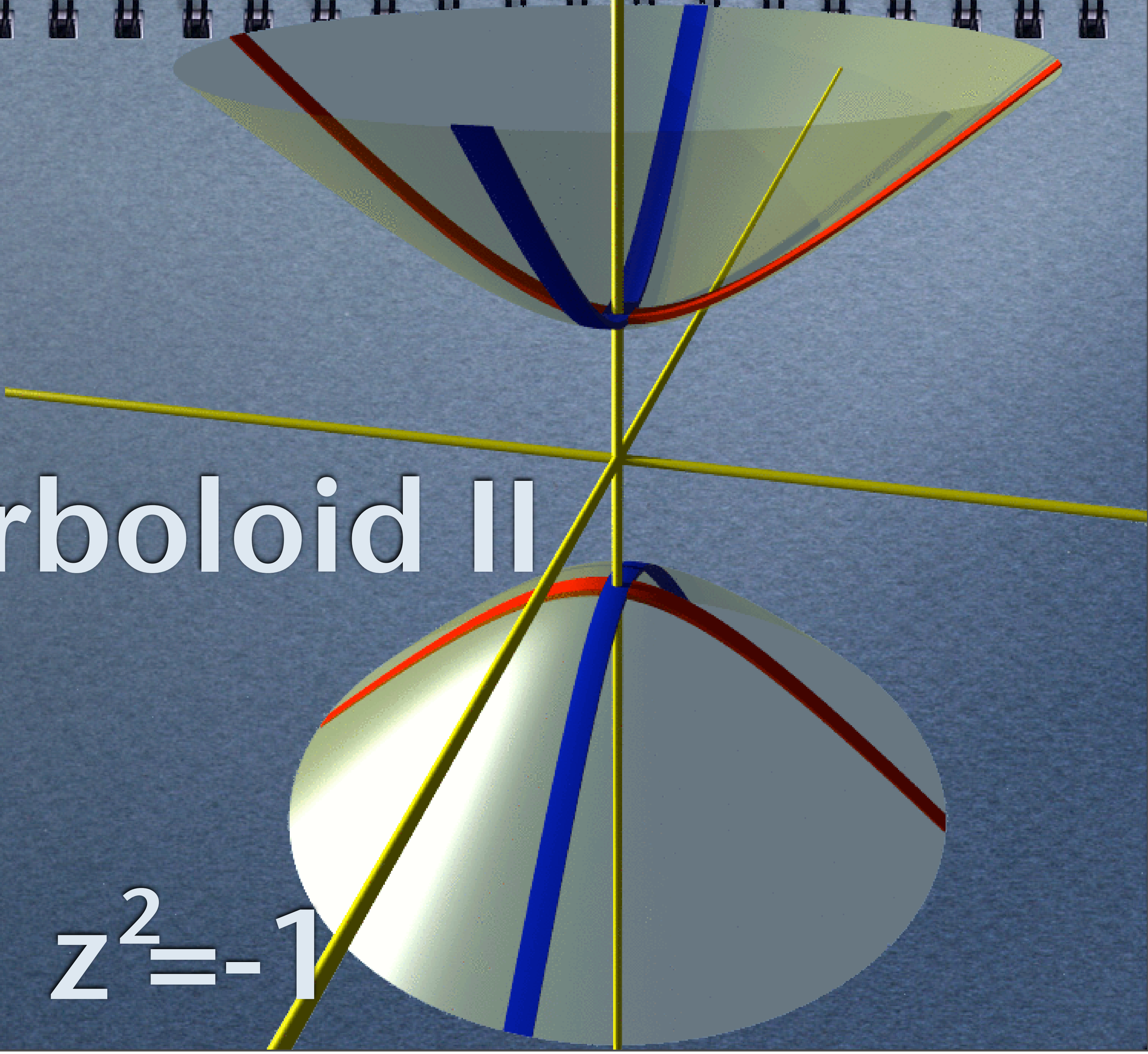
$$x^2 + y^2 + z^2 = 1$$

Hyperboloid I



$$x^2 + y^2 - z^2 = 1$$

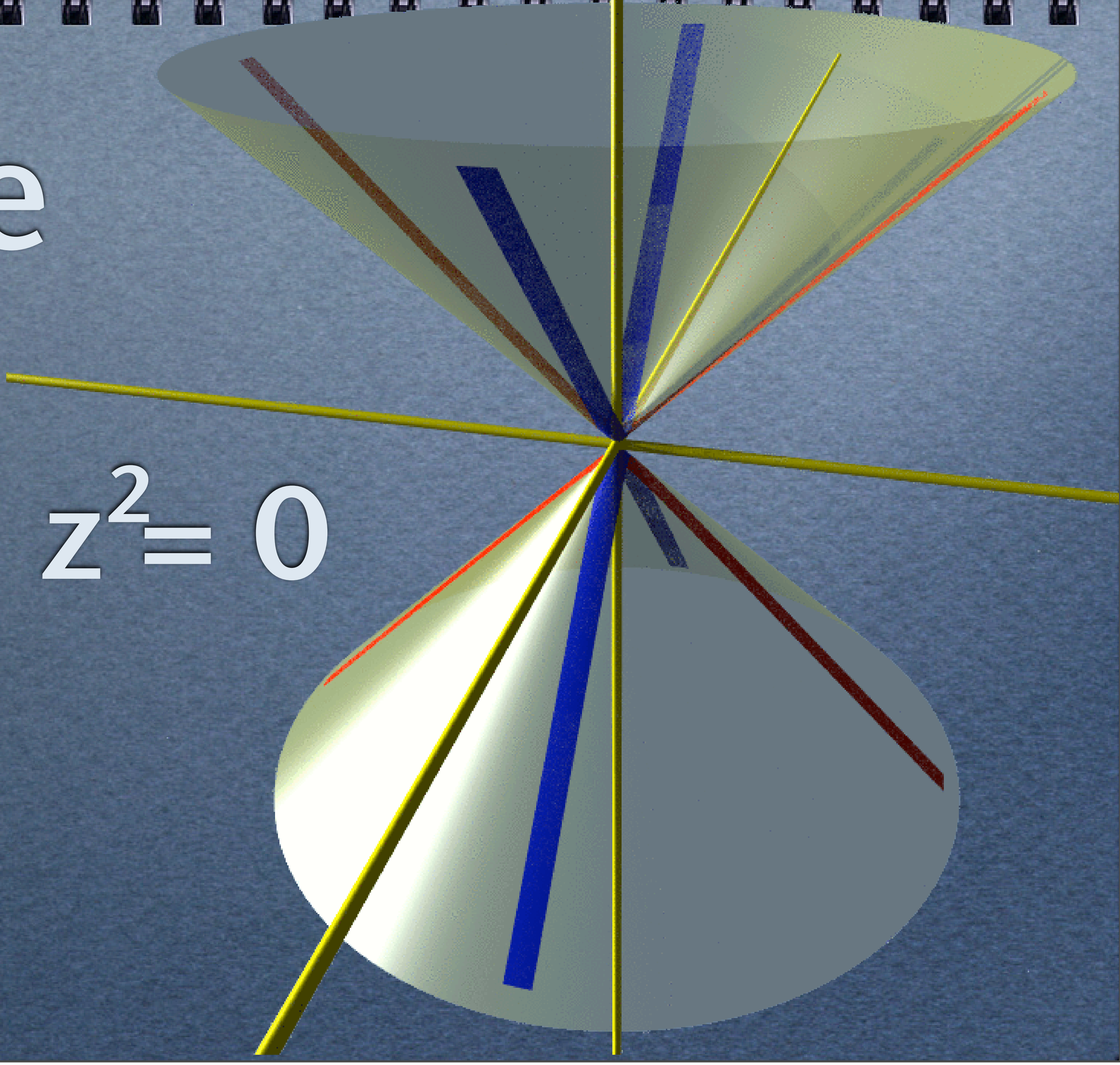
Hyperboloid II



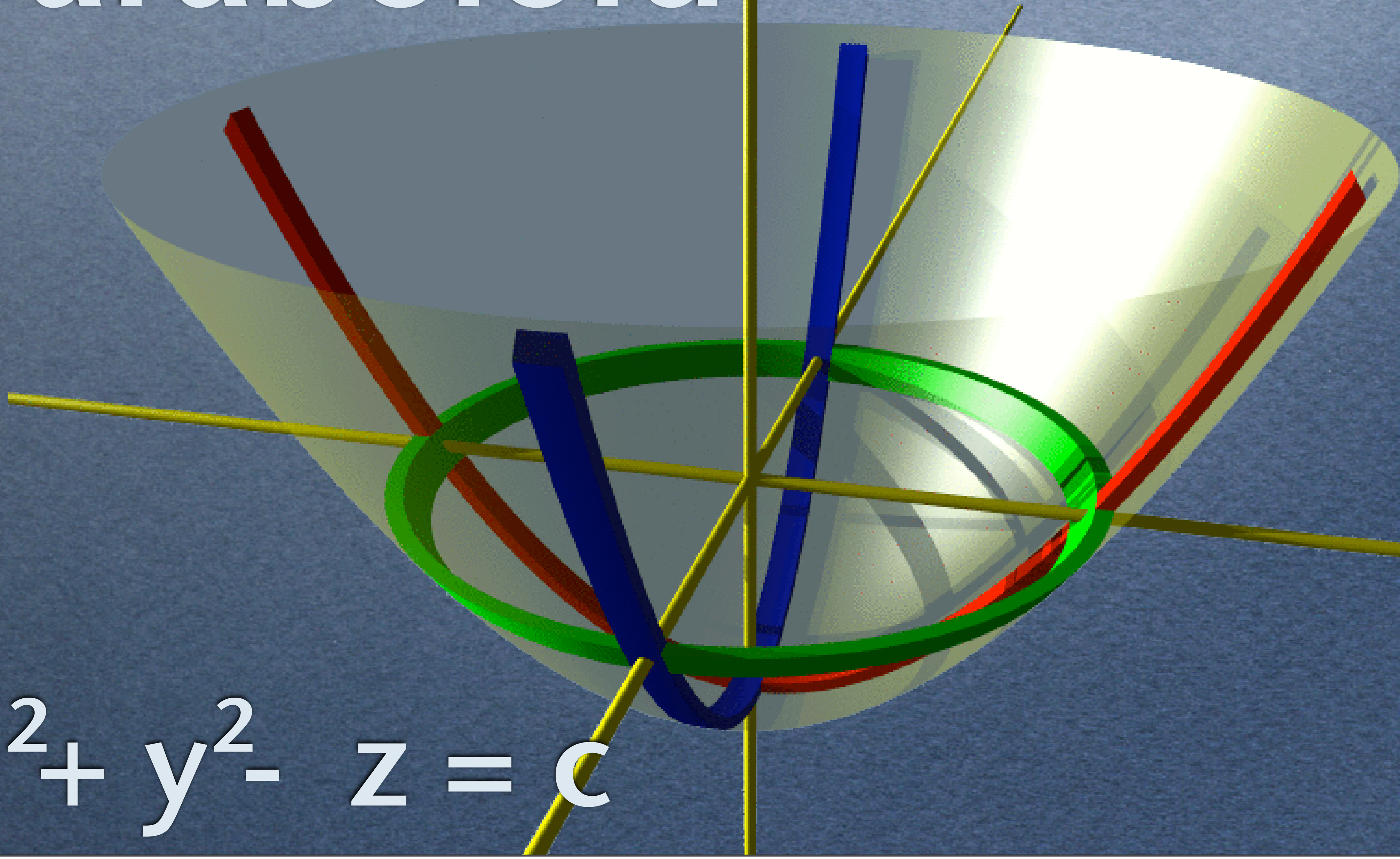
$x^2 + y^2 - z^2 = -1$

Cone

$$x^2 + y^2 - z^2 = 0$$

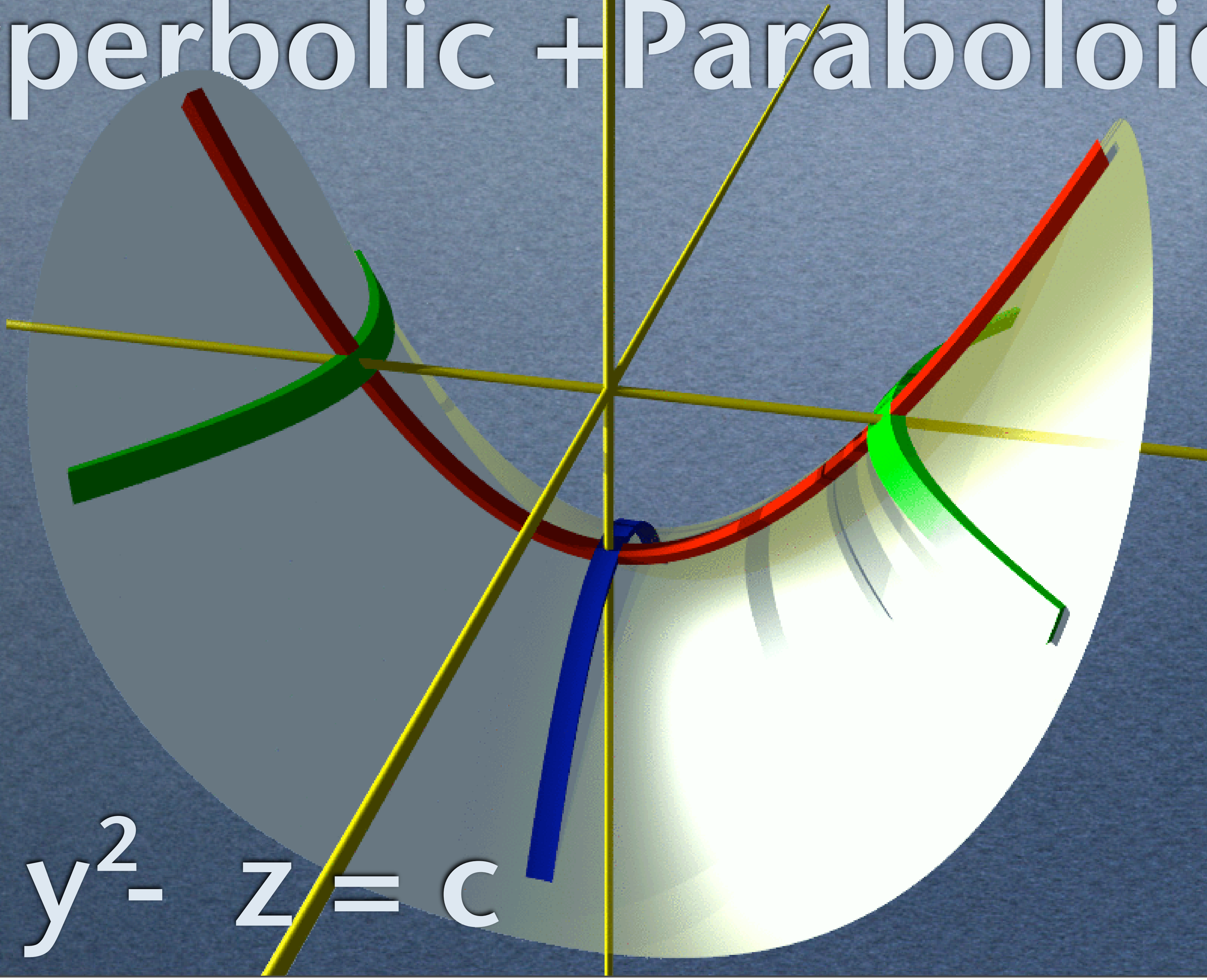


Paraboloid



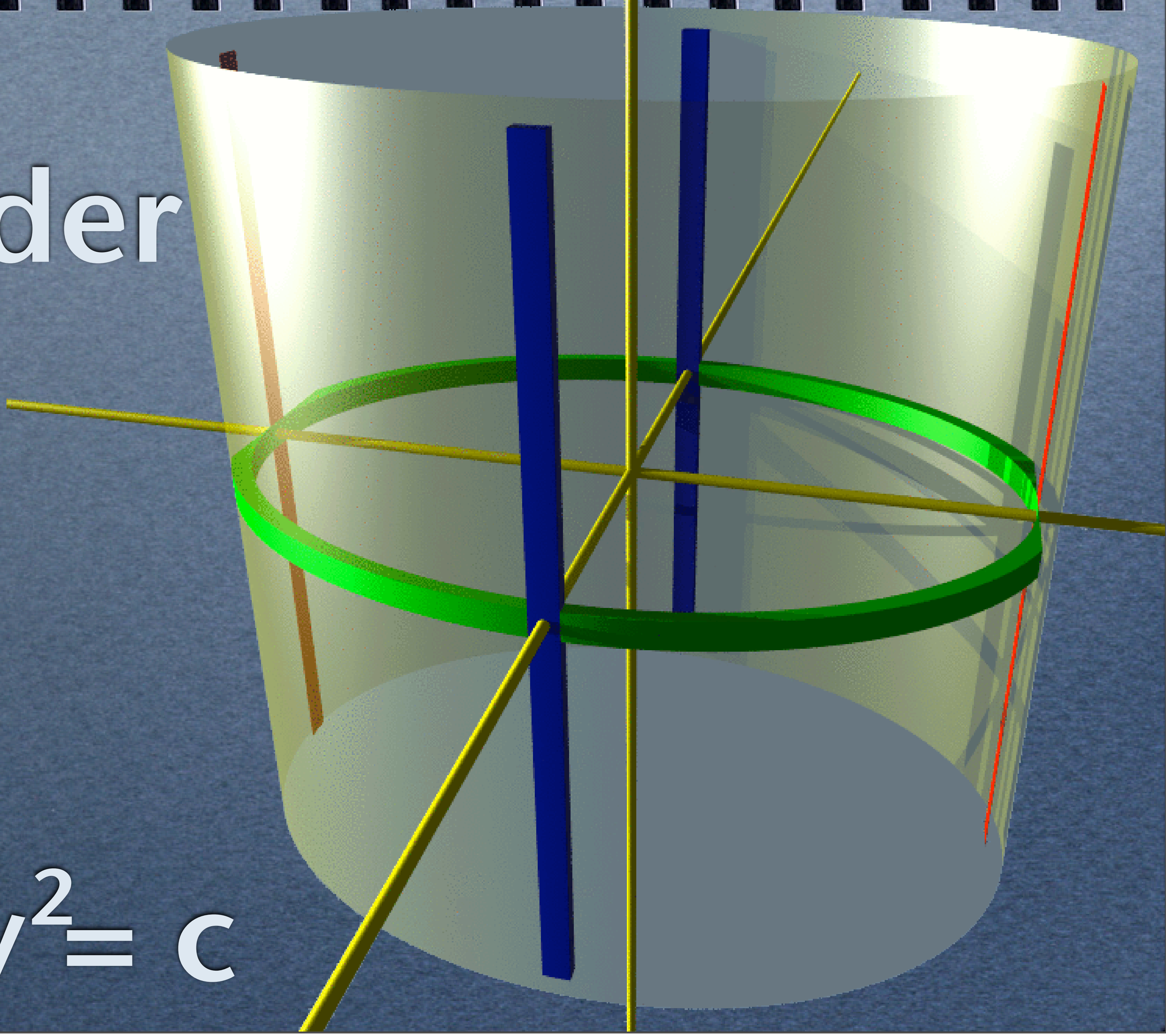
$$x^2 + y^2 - z = c$$

Hyperbolic Paraboloid



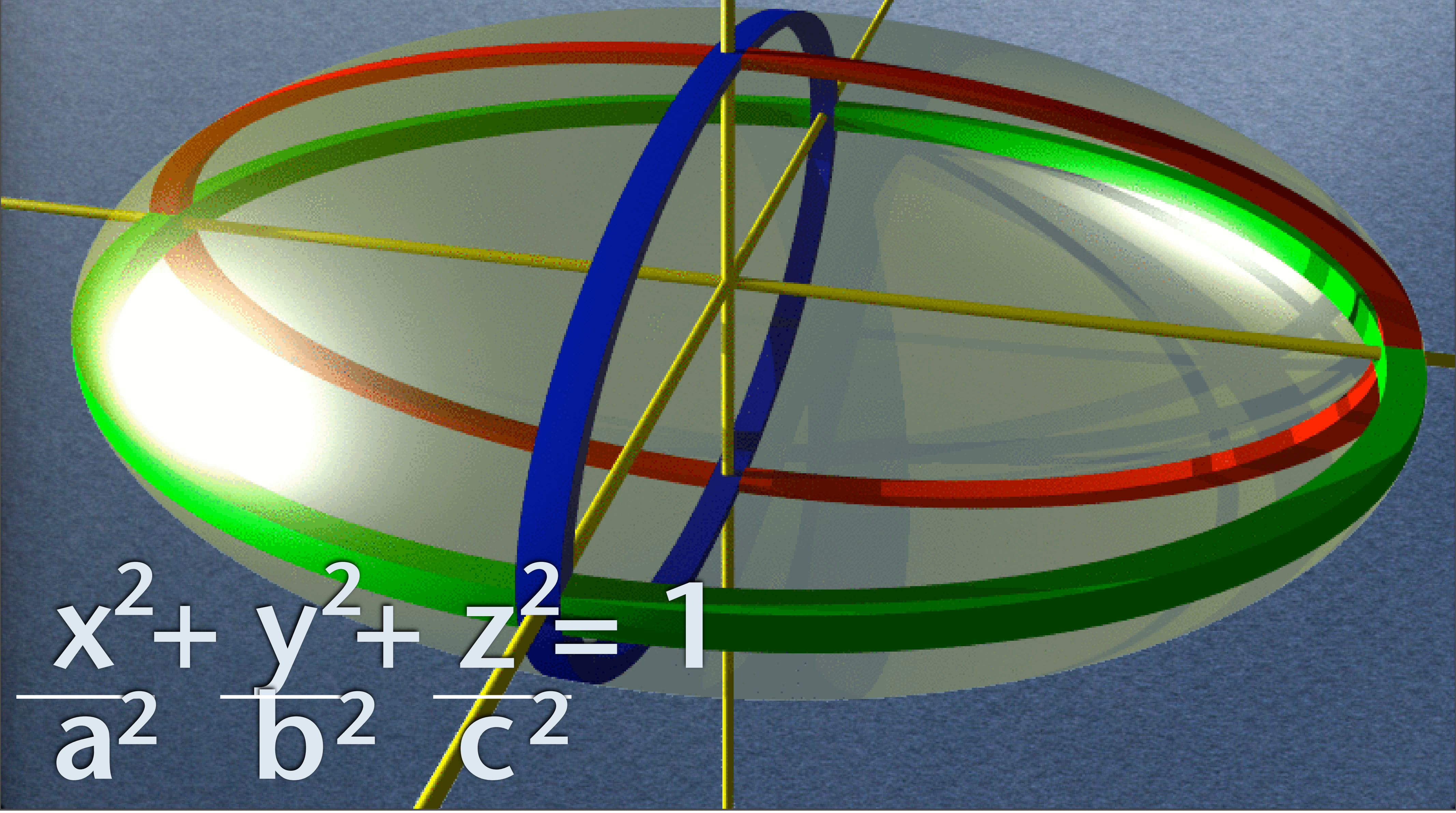
$x^2 - y^2 - z = c$

Cylinder



$x^2 + y^2 = c$

Ellipsoid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

How to analyze ?

- Traces
- Check with known surfaces

What surface is:

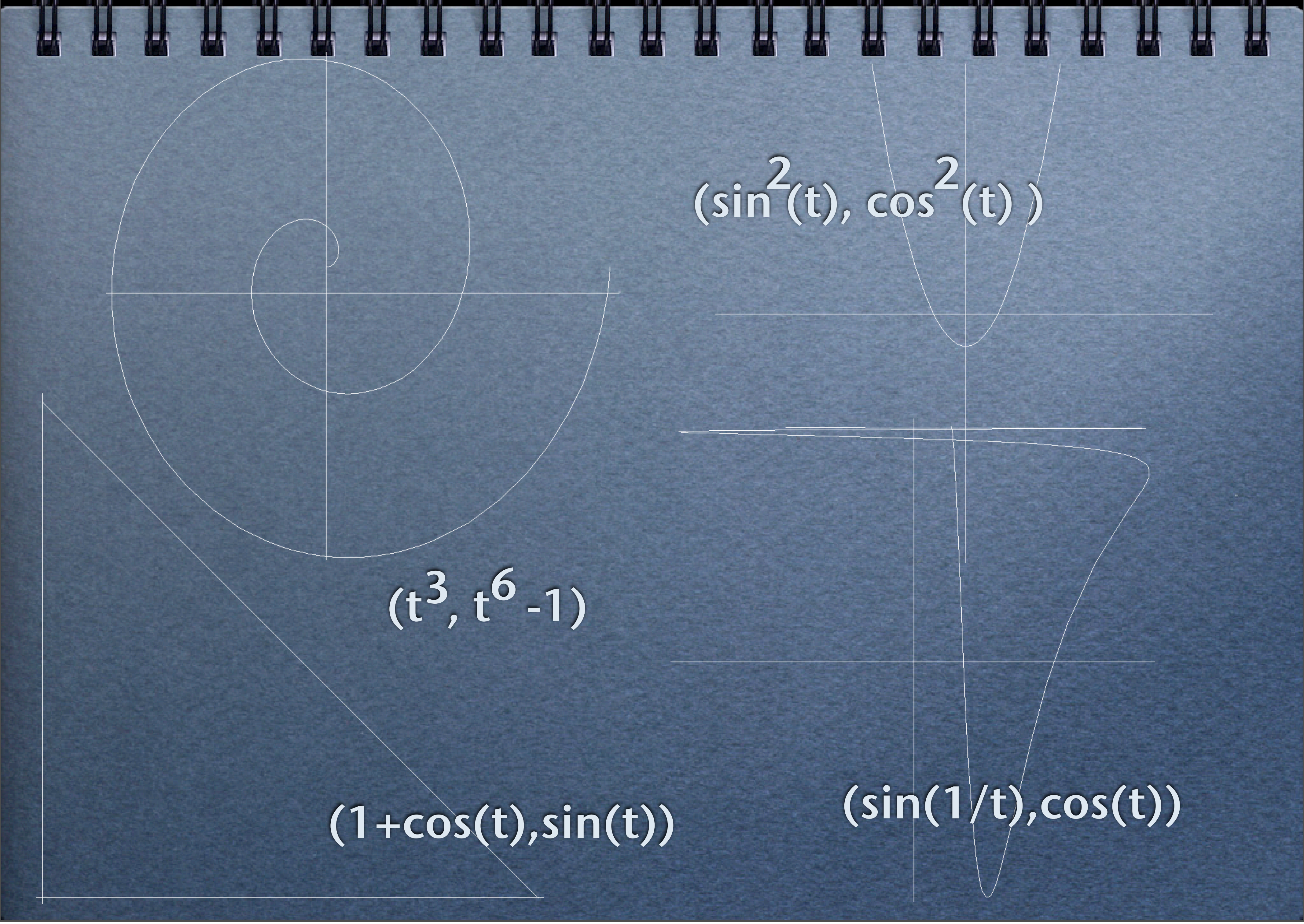
$$x^2 + y^2 - z^2 = 2z$$

Matching problems

Eliminate easy ones.

Symmetries.

Relations.


(sin2(t), cos2(t))
(t3, t6 -1)
(1+cos(t),sin(t))
(sin(1/t),cos(t))

A last remark:

We were often asked questions like: What topics will be covered most in the exam.

I can say so much:

The exam will not look as the following exam,which had once given at the Math department by David Mumford:

1: Write an exam for this course

2: Take it

Steven Krantz mentions in one of his Apocrypha books the rumor that the Harvard philosophy department had once a similar exam, but with a third question:

3:Grade it

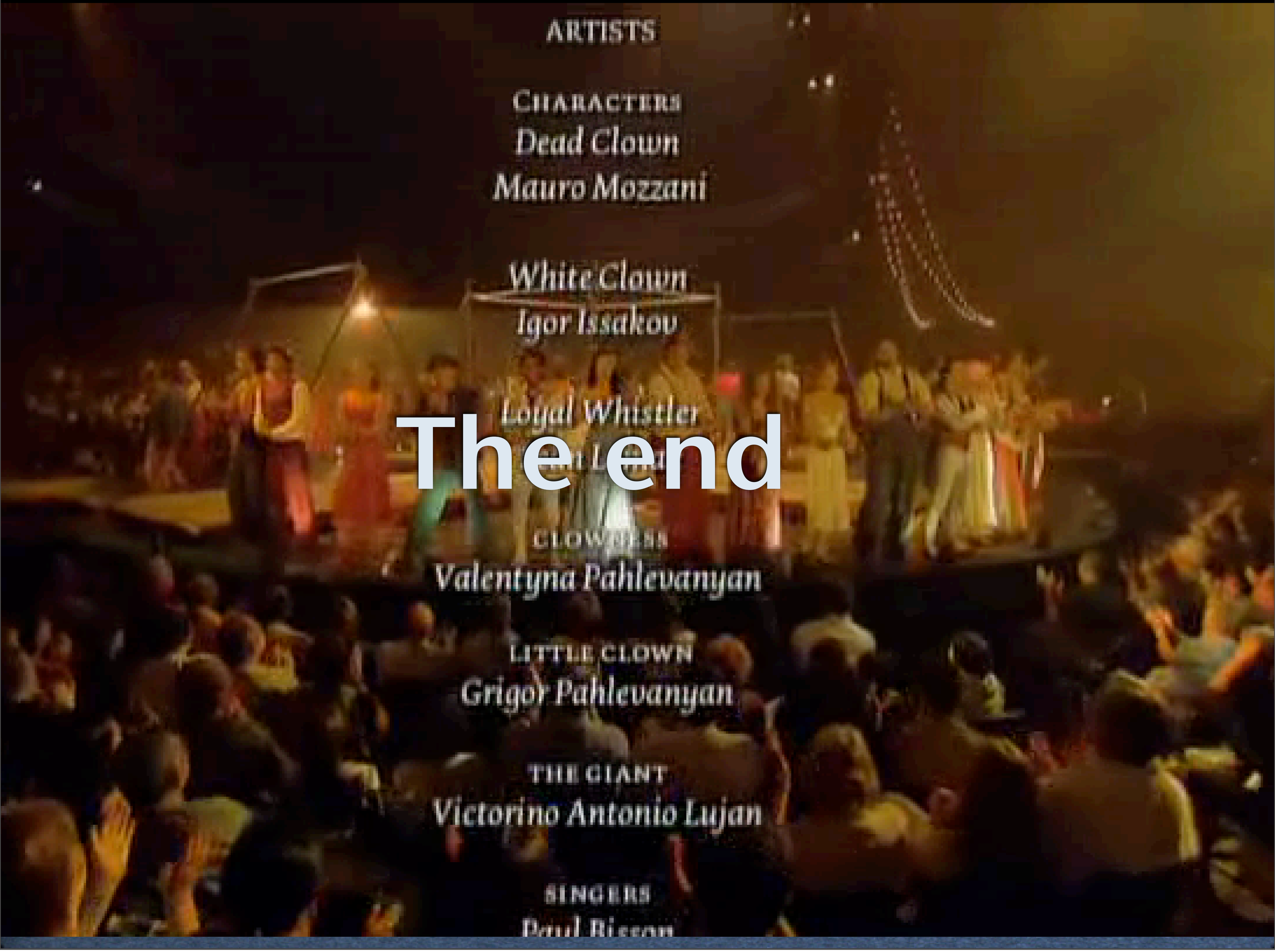
So:

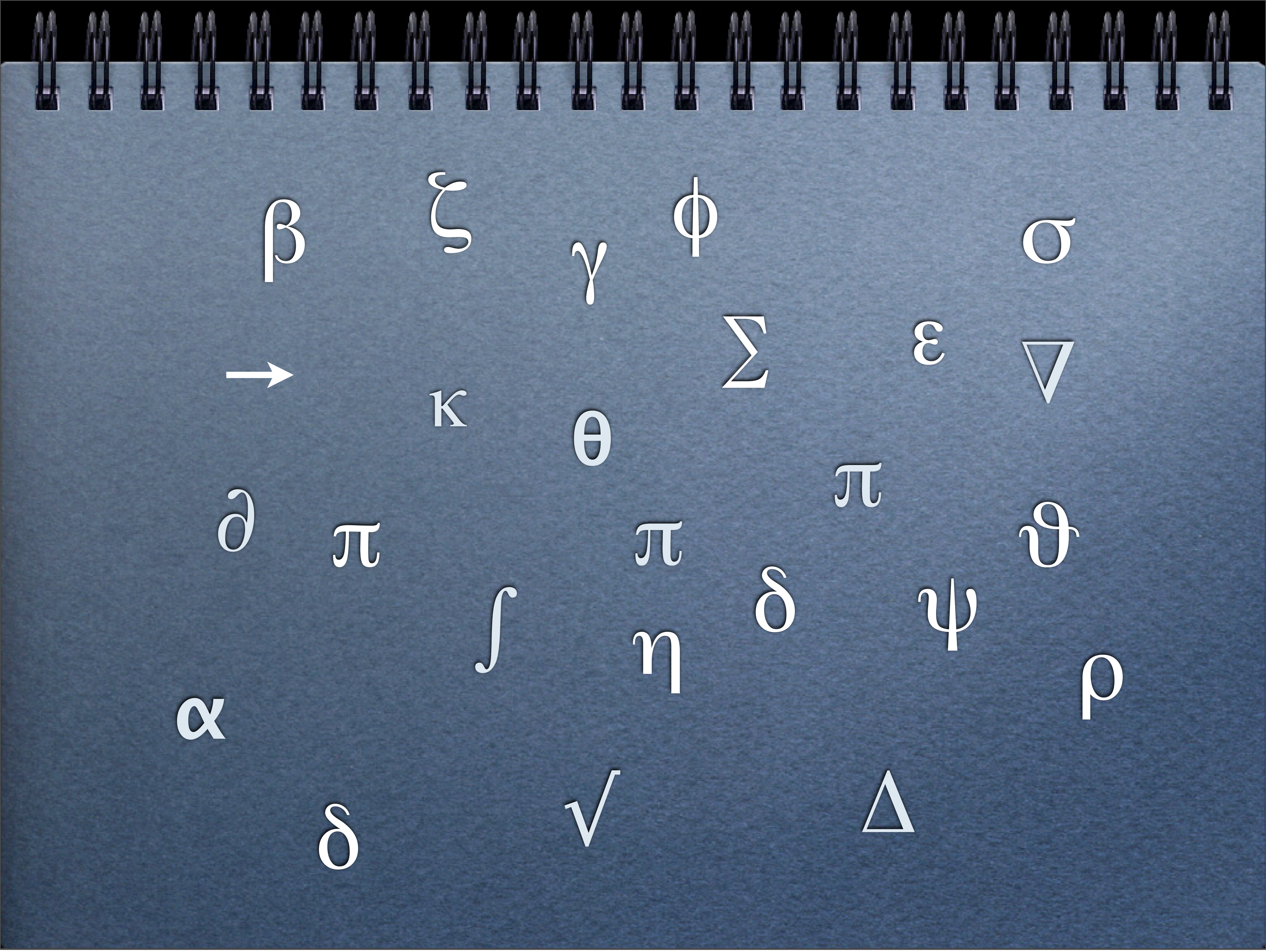
While we certainly would love to do that too, let me assure you that this is not the type of exams, you will encounter next Wednesday.

Questions?

I will take first a few questions in plenum, then stay available for individual questions.

The end


ARTISTS
CHARACTERS
Dead Clown
Mauro Mozzani
White Clown
Igor Issakov
Loyal Whistler
Valentyna Pahlevanyan
LITTLE CLOWN
Grigor Pahlevanyan
THE GIANT
Victorino Antonio Lujan
SINGERS


β ζ γ ϕ σ
Σ ε ∇
→ κ θ
π
∂ π π ϑ
δ ψ
∫ η
ρ
α
√ Δ
δ