Math 21A - supplement to the handout "distances"

Sug Woo Shin

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Case 1: Distance from a point to a plane

Let P' be the projected image of P onto the plane Σ. Let $\theta = \angle QPP'$. Then

$$d(P,\Sigma) = \|\overrightarrow{PP'}\| = \|\overrightarrow{PQ}\||\cos\theta| = \frac{|\overrightarrow{PQ}\cdot\vec{n}|}{\|\vec{n}\|}$$

Case 2: Distance from a point to a line

Let P' be the projected image of P onto the line L. Let $\theta = \angle PQP'$. Then

$$d(P,L) = \|\overrightarrow{PP'}\| = \|\overrightarrow{PQ}\|\sin\theta = \frac{\|\overrightarrow{PQ}\times\vec{u}\|}{\|\vec{u}\|}$$

Case 3: Distance from a line to another line

(1) If the lines L and M are parallel, then $\vec{u}\times\vec{v} = 0$, so the formula in the handout doesn't make sense. In this case, you know that

$$d(L,M) = d(Q,M)$$

where Q is a point on L. So you appeal to Case 2.

(2) If L and M are not parallel, choose a line M' which is parallel to M and intersects with L. Let Σ be the plane containing L and M'. Then

$$d(L, M) = d(M, \Sigma) \overset{(a)}{=} d(P, \Sigma) \overset{(b)}{=} \frac{|\overrightarrow{PQ} \cdot (\vec{u} \times \vec{v})|}{\|\vec{u} \times \vec{v}\|}$$

where (a) is true because M is parallel to Σ and (b) follows from Case 2, since a normal vector of the plane Σ is given by the cross product $\vec{u} \times \vec{v}$.

Case 4: Distance from a plane to another plane

Any two non-parallel planes in the 3-D space intersect, in which case the distance is 0. Now assume that two planes Σ, Π are parallel, in which case their equations may be written using the same normal vector $\vec{n}$. Say the planes Σ, Π are given by the equations $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$, respectively. Choose a point P in Σ and a point Q in Π. Then

$$d(\Sigma, \Pi) = d(P, \Pi) \overset{(*)}{=} \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|e - d|}{\|\vec{n}\|}$$

where $(*)$ follows from Case 1. The last equality is true since

$$\overrightarrow{PQ} \cdot \vec{n} = (\overrightarrow{OQ} - \overrightarrow{OP}) \cdot \vec{n} = \overrightarrow{OQ} \cdot \vec{n} - \overrightarrow{OP} \cdot \vec{n} = e - d$$