Math 21A (Sug Woo Shin) - Mass and Moments of Inertia - 11/22

Notions

• **center of mass**: conceptually "weighted average of positions of mass".

(ex. If you have mass m_1 at (x_1, y_1) and m_2 at (x_2, y_2),
then the center of mass is $\left(\frac{m_1x_1+m_2x_2}{m_1+m_2}, \frac{m_1y_1+m_2y_2}{m_1+m_2}\right)$.)

Remember (each coordinate of center of mass)=(1st moment) $\div$ (total mass)

• **moment of inertia**: always taken *with respect to an axis of rotation.*

(moment of inertia)=(distance from axis)^2 $\times$ (mass)

	discrete world	continuous world
sum	$\sum$	$\int$
distr.of mass	mass m_i at (x_i, y_i)	density function $\delta(x, y)$
total mass	$M = \sum_i m_i$	$M = \int\int_R \delta(x, y)\ dA$
1st moments	$M_{x=0} = \sum_i m_i x_i$, $M_{y=0} = \sum_i m_i y_i$	$M_{x=0} = \int\int_R x\delta(x, y)\ dA$, $M_{y=0} = \int\int_R y\delta(x, y)\ dA$
center of mass	$\left(\frac{M_{x=0}}{M}, \frac{M_{y=0}}{M}\right)$	$\left(\frac{M_{x=0}}{M}, \frac{M_{y=0}}{M}\right)$
moments of inertia	$I_{x=0} = \sum_i m_i x_i^2$, $I_{y=0} = \sum_i m_i y_i^2$	$I_{x=0} = \int\int_R x^2\delta(x, y)\ dA$, $I_{y=0} = \int\int_R y^2\delta(x, y)\ dA$

Remark 1. (3-dimensional version)

Generalization to 3-dim space is straightforward (except for the moments of inertia):
add z-compo., two var. $\Rightarrow$ three var., double integrals $\Rightarrow$ triple integrals.

What about the <u>moments of inertia in 3-D</u>? We take them with respect to x, y, z-axes.

$I_{x-axis} = \int\int\int_U (y^2 + z^2)\ \delta(x, y, z)\ dV,$
$I_{y-axis} = \int\int\int_U (x^2 + z^2)\ \delta(x, y, z)\ dV,$
$I_{z-axis} = \int\int\int_U (x^2 + y^2)\ \delta(x, y, z)\ dV$

Note that $y^2 + z^2$ is the (distance)^2 of the point (x, y, z) from x-axis, etc.

Exercise 2. The density $\delta(x, y, z) = y - x$ is given over the solid

$$U = \{(x, y, z) : 1 \leq z \leq 2,\ 0 \leq x \leq z,\ x \leq y \leq z\}.$$

Find (1) the mass, (2) the three first moments, (3) the center of mass, and (4) the moments of inertia with respect to all three axes.

Exercise 3. (Quiz)

We have (1) a hollow ball, (2) solid ball, and (3) a point mass having the *same mass* and the *same center*. Compare their first moments and the moments of inertia (say, with respect to z-axis).

Probability, expectation, variance

If you roll a dice, you get the number X (called "**random variable**").

$P(X = i) = 1/6, \qquad i = 1, 2, 3, 4, 5, 6.$

The numbers $P(X = i)$ account for the probability distribution of X. Of course, if your dice is not fair, you may get different numbers for $P(X = i)$, but it is still true that $\sum_{i=1}^{6} P(X = i) = 1$.

Now the expectation for X is $E(X) = \sum_{i=1}^{6} P(X = i) \times i$.

This is a discrete 1-dimensional version of our physical notion:

"mass" $\Leftrightarrow P(X = i)$, $i = 1, ..., 6$.

"total mass"=1. (That's probability.)

"center of mass"=expectation.

Let's do this in the continuous 2-variable case.

Now $\delta(x, y)$ is a **probability density function** (p.d.f.), satisfying

$(\delta(x, y) \geq 0, \int\int_{xy-plane} \delta(x, y)\ dA = 1.)$

$\delta(x, y)$ gives the distribution of random variable (X, Y).

$P(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b \delta(x, y) dx dy$

Example: Suppose you fire the cannon, aiming at (0,0). The probability distribution for hitting the location (x, y) is expressed by $\delta(x, y)$.

Now the "center of mass" yields $E(X, Y)$.

Question: what about the *moments of inertia*?

It's a kind of variance in statistics. The moment of inertia with respect to (the line given by) $x = E(X)$ gives the variance $V(X)$. The moment of inertia with respect to $y = E(Y)$ gives the variance $V(Y)$.

(This doesn't work well for three variable case.)