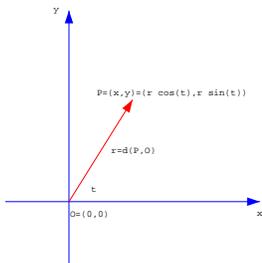


**POLAR INTEGRATION**

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HOMEWORK: 14.5: 8,10, 30,54,60

REMINDER: POLAR COORDINATES. A point  $(x, y)$  in the plane has the **polar coordinates**  $r = \sqrt{x^2 + y^2}, \theta = \arctg(y/x)$ . We have  $x = r \cos(\theta), y = r \sin(\theta)$ .



POLAR CURVES. A general polar curve is written as  $(r(t), \theta(t))$ . It can be translated into  $x, y$  coordinates:  $x(t) = r(t) \cos(\theta(t)), y(t) = r(t) \sin(\theta(t))$ .

POLAR GRAPHS. Curves which are graphs when written in polar coordinates are called **polar graphs**.

EXAMPLE.  $r(\theta) = \cos(3\theta)$  is the which belongs to the class of **roses**  $r(t) = \cos(nt)$ .

EXAMPLE. If  $y = 2x + 3$  is a line, then the equation gives  $r \sin(\theta) = 2r \cos(\theta) + 3$ . Solving for  $r(t)$  gives  $r(\theta) = 3/(\sin(\theta) - 2 \cos(\theta))$ . The line is also a polar graph.

EXAMPLE. The polar form  $r(\theta) = \frac{a(1-e^2)}{1+e \cos(\theta)}$  of the ellipse (see Kepler). The ellipse is a polar graph.

INTEGRATION IN POLAR COORDINATES. For many regions, it is better to use polar coordinates for integration:

$$\int \int f(x, y) dx dy = \int \int g(r, \theta) r dr d\theta$$

For example if  $f(x, y) = x^2 + x^2 + xy$ , then  $g(r, \theta) = r^2 + r^2 \cos(\theta) \sin(\theta)$ .

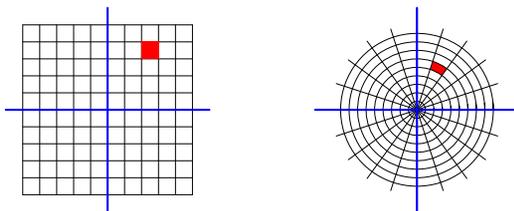
EXAMPLE. We had computed area of the disc  $\{x^2 + y^2 \leq 1\}$  using substitution. Was quite a mess. It is easier to do that integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r dr d\theta = 2\pi r^2/2|_0^1 = \pi.$$

WHERE DOES THE FACTOR "r" COME FROM?

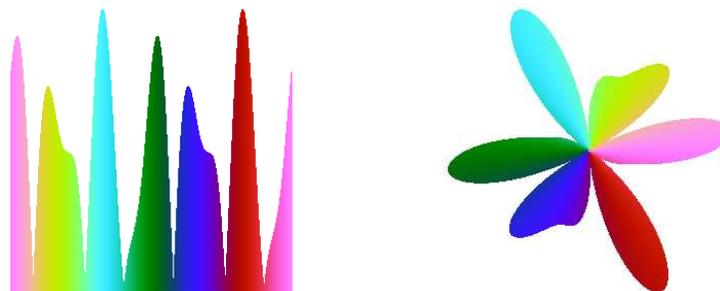
A small rectangle  $R$  with dimensions  $d\theta dr$  in the  $(r, \theta)$  plane is mapped by  $T : (r, \theta)$

$\mapsto$   
 $(r \cos(\theta), r \sin(\theta))$  to a sector segment  $S$  in the  $(x, y)$  plane. It has approximately the area  $r d\theta dr$ . It is small for small  $r$ .



AN OTHER EXPLANATION. For fixed  $r$ , the circle  $\vec{r}(\theta) = T(r, \theta) = (r \cos(\theta), r \sin(\theta))$  has the velocity vector  $\vec{v} = (v_1, v_2) = (-r \sin(\theta), r \cos(\theta))$ . For fixed  $\theta$ , the line  $\vec{r}(r) = (r \cos(\theta), r \sin(\theta))$  has the velocity vector  $\vec{w} = (w_1, w_2) = (\cos(\theta), \sin(\theta))$ . The parallelogram spanned by the vectors has the area  $|\vec{v} \times \vec{w}| = v_1 w_2 - w_1 v_2 = r$ .

ROSES. Examples of type I regions in the  $(\theta, r)$  plane are **roses**:  $\{(\theta, r) | 0 \leq r \leq f(\theta)\}$  where  $f(\theta)$  is a periodic function of  $\theta$ .



The region  $R$  in the  $\theta - r$  coordinates is a type I region

The region  $S = T(R)$  in the  $x - y$  coordinates is neither a type I nor a type II region.

EXAMPLE. Find the area of the region  $\{(\theta, r(\theta)) | r(\theta) \leq |\cos(3\theta)|\}$ .

$$\int \int_R y dx dy = \int_0^{2\pi} \int_0^{|\cos(3\theta)|} r dr d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2} d\theta = \pi/2$$

EXAMPLE. Integrate  $f(x, y) = y\sqrt{x^2 + y^2}$  over the semi annulus  $R = \{(x, y) | 1 < x^2 + y^2 < 4, y > 0\}$ .

Solution.

$$\int_1^2 \int_0^\pi r \sin(\theta) r r d\theta dr = \int_1^2 r^3 \int_0^\pi \sin(\theta) d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^\pi \sin(\theta) d\theta = 15/2$$

For integration problems, where the region is part of an annulus, or if you see function with terms  $x^2 + y^2$  try to use polar coordinates  $x = r \cos(\theta), y = r \sin(\theta)$ .

THE SUPER-CURVE. The Belgian Biologist Johan Gielis came up in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left( \frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b} \right)^{-1/n_3}$$

```
S[m_,n1_,n2_,n3_,a_,b_] := Module[{t},
  r[t_] := Abs[Cos[t/4]/a]^n1;
  r[t_] := Abs[Sin[t/4]/b]^n2;
  R[t_] := (r[t]^n1 + r[t]^n2)^(1/n3);
  f[t_] := If[R[t] == 0, {0, 0}, {Cos[t], Sin[t]}/R[t]];
  ParametricPlot[r[t], {t, -2Pi, 2Pi}, PlotRange -> All,
  PlotPoints -> 1000, Axes -> False, AspectRatio -> 1, Frame -> False];
];
RI = Random[Integer, {3, 20}]; RR = 20 Random[];
T = S[RI, RR, RR, RR, RR, RR]
```

It is called the **super-curve** because it can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes" (see later).

The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to tackle one of the more intractable problems in biology: describing form. A twist: Gielis has patented his discovery described in "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003). To the right you see the Mathematica code.

