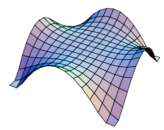
HOMEWORK: 13.3: 10, 14. 13.4: 20, 32, 50

PARTIAL DERIVATIVE. If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y,z)$ is defined as the derivative of the function $g(x) = f(x,y,z)$, where y is fixed. The partial derivative with respect to y is defined similarly.

NOTATION. One also writes $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$ etc. For iterated derivatives the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$.

REMARK. The notation for partial derivatives $\partial_x f, \partial_y f$ were introduced by Jacobi. Lagrange had used the term "partial differences". Partial derivatives measure the rate of change of the function in the x or y directions. For functions of more variables, the partial derivatives are defined in a similar way.

EXAMPLE. $f(x,y) = x^4 - 6x^2y^2 + y^4$. We have $f_x(x,y) = 4x^3 - 12xy^2, f_{xx} = 12x^2 - 12y^2, f_y(x,y) = -12x^2y + 4y^3, f_{yy} = -12x^2 + 12y^2$. We see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation**.



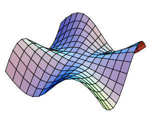
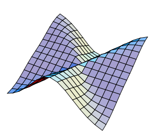
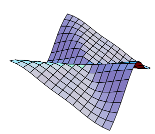
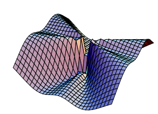
CLAIROT THEOREM. If f_{xy} and f_{yx} are both continuous, then $\boxed{f_{xy} = f_{yx}}$. Proof. Compare the two sides:

$dx f_x(x,y) \sim f(x+dx,y) - f(x,y)$ $dy dx f_{xy}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y+dy) - (f(x+dx,y) - f(x,y))$	$dy f_y(x,y) \sim f(x,y+dy) - f(x,y)$. $dx dy f_{yx}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y) - (f(x,y+dy) - f(x,y))$

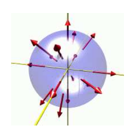
CONTINUITY IS NECESSARY. Example: $f(x,y) = (x^3y - xy^3)/(x^2+y^2)$ contradicts Clairot:

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,	$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

$\boxed{f(x,y)}$ $\boxed{f_x(x,y)}$ $\boxed{f_y(x,y)}$ $\boxed{f_{xy}(x,y)}$

PREVIEW: As we will see later, the gradient $\nabla f(x,y)$ is orthogonal to the level curve $f(x,y) = c$ and the gradient $\nabla f(x,y,z)$ is normal to the level surface $f(x,y,z)$. For example, the gradient of $f(x,y,z) = x^2 + y^2 - z^2$ at a point (x,y,z) is $(2x, 2y, -2z)$.



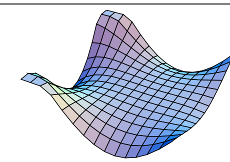
- The geometric significance will be seen later. We will be able to construct vectors normal to level curves or level surfaces.
- Partial derivatives will help us to find approximations to surfaces, linarizations to nonlinear functions.
- Partial differential equations are laws describing nature. They involve functions of several variables and derivatives with respect to several veriables.
- In a bit more than a week, we will solve optimization problems for functions in several variables.
- Towards the end of the course, we will find solution to some integration problems using generalizations of the fundamental theorem of calculus.
- Partial derivatives are in general helpful to understand and analyze functions of several variables.

PARTIAL DIFFERENTIAL EQUATIONS. An equation which involves partial derivatives of an unknown function is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

1) $f_{xx}(x,y) = f_{yy}(x,y)$ is an example of a partial differential equation 1) $f_x(x,y) = f_{xx}(x,y)$ is an example of an ordinary differential equation. The variable y can be considered as a parameter.

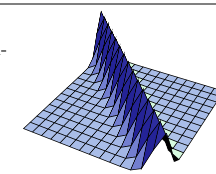
LAPLACE EQUATION

$\boxed{f_{xx} + f_{yy} = 0}$.
$f(x,t) = x^2 - y^2$.



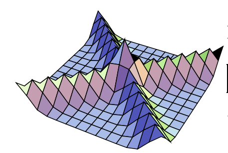
ADVECTION EQUATION

$\boxed{f_t = f_x}$.
$f(x,t) = g(x-t)$.



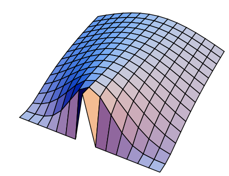
WAVE EQUATION

$\boxed{f_{tt} = f_{xx}}$. $f(t,x) = \sin(x-t) + \sin(x+t)$



HEAT EQUATION

$\boxed{f_t = f_{xx}}$
$f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$



"A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." - Paul A. M. Dirac.

Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpretated the later as an **antiparticle**: the existence of antiparticles was later confirmed.

PROBLEM: Verify that $f(x,t) = e^{-rt}\sin(x+ct)$ satisfies the advection equation $f_t(x,t) = cf_x(x,t) - rf(x,t)$.