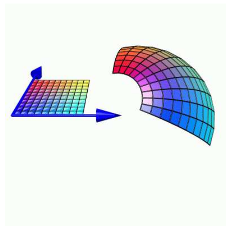
PARAMETRIC SURFACES	Math21a, O. Knill

HW A) Parametrize the upper half of the ellipsoid $x^2 + y^2 + z^2/4 = 1$ in three different ways: as a graph $\vec{r}(x, y) = (x, y, f(x, y))$ (Euclidean coordinates), as a surface of revolution $\vec{r}(\theta, z) = (g(z)\cos(\theta), g(z)\sin(\theta), z)$ (cylindrical coordinates) and as a deformed sphere $\vec{r}(\phi, \theta) = (x(\phi, \theta), y(\phi, \theta), z(\phi, \theta))$ (spherical coordinates).
B) The curve $\vec{r}(t) = (\sqrt{t}\cos(t), \sqrt{t}\sin(t), t)$ is on a surface. Find a parametrization $\vec{r}(t, s)$ of this surface.
C) Parametrize the surface which has distance 1 from the unit circle in the xy plane. This is a doughnut. Use two angles θ, a rotation angle around the z axes and ϕ a rotation angle around the circle.
D) Parametrize the paraboloid $x^2 + y^2 = z$ in two different ways: as a graph or as a surface of revolution.
E) Upload a picture for a gallery of marbles.

PARAMETRIC SURFACES. The image of a map

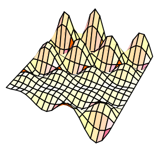
$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

defines a **surface**. r is called the **parametrisation** of the surface. The surface is defined by three functions $x(u, v), y(u, v), z(u, v)$ each is a function of two variables. If we fix one of the variables, say $v = v_0$, then $u \mapsto \vec{r}(u, v_0)$ is a curve on the surface. Similarly, if we fix $u = u_0$, then $v \mapsto \vec{r}(u_0, v)$ is a curve on the surface.



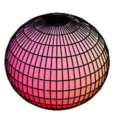
GRID CURVES. If we keep u constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u, v)$ is a curve on the surface. These curves are called **grid curves**. If you plot a surface with a computer, the pictures usually show these grid curves.

EXAMPLES. The world of parametric surfaces is fantastic. You will explore this land with the help of the computer algebra system Mathematica. Here first a few examples:



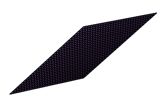
Graphs

$$\begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$$



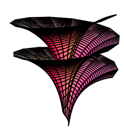
Sphere

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{bmatrix}$$



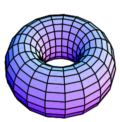
Plane through points P, Q, R

$$\begin{matrix} P+ \\ u(Q-P)+ \\ v(R-P) \end{matrix}$$



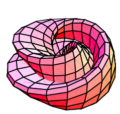
Dini surface

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) + \log(\tan(\frac{v}{2})) + \frac{u}{5} \end{bmatrix}$$



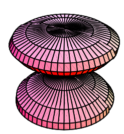
Torus

This is a homework



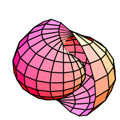
Klein bottle

$$\begin{bmatrix} (\frac{3}{2} + \cos(u)\sin(2v) - \sin(u)\sin(4v))\cos(2u) \\ (\frac{3}{2} + \cos(u)\sin(2v) - \sin(u)\sin(4v))\sin(2u) \\ \frac{3}{4}\sin(u)\sin(2v) + \cos(u)\sin(4v) \end{bmatrix}$$



Eight surface

$$\begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) + \log(\tan(\frac{v}{2})) + \frac{u}{5} \end{bmatrix}$$



Snail surface

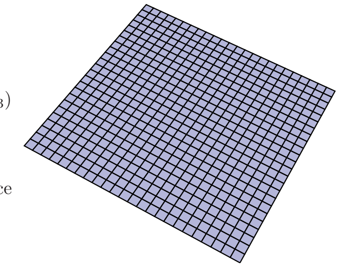
$$\begin{bmatrix} \cos(u)\sin(2v) \\ \sin(u)\sin(2v) \\ \sin(v) \end{bmatrix}$$

4 BASIC EXAMPLES (THESE ARE THE ONES TO KNOW)

PLANE.

Parametric: $r(u, v) = (P_1 + uu_1 + vv_1, P_2 + uu_2 + vv_2, P_3 + uu_3 + vv_3)$
Implicit: $ax + by + cz = d$.

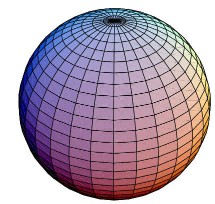
Parametric to Implicit: $(a, b, c) = (u_1, u_2, u_3) \times (v_1, v_2, v_3)$.
Implicit to Parametric: Find three points P, Q, R on the surface and form $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.



SPHERE.

Parametric: $r(u, v) = (\rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v))$
Implicit: $x^2 + y^2 + z^2 = \rho^2$.

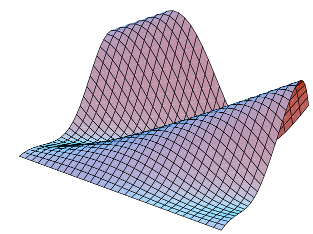
Parametric to Implicit: read of radius.
Implicit to Parametric: know it



GRAPH.

Parametric: $r(u, v) = (u, v, f(u, v))$
Implicit: $z - f(x, y) = 0$.

Parametric to Implicit: think about $z = f(x, y)$
Implicit to Parametric: use x and y as the parameterizations.



SURFACE OF REVOLUTION.

Parametric: $r(u, v) = (g(v)\cos(u), g(v)\sin(u), v)$
Implicit: $\sqrt{x^2 + y^2} = r = g(z)$ can be written as $x^2 + y^2 = g(z)^2$.

Parametric to Implicit: read off the function $g(z)$ the distance to the z-axis.
Implicit to Parametric: use the function g.