

## ARC LENGTH

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HOMEWORK: Section 12.3: 12,14,42, 56, 58

### PLANE CURVE

$\vec{r}(t) = \langle x(t), y(t) \rangle$  **position**  
 $\vec{r}'(t) = \langle x'(t), y'(t) \rangle$  **velocity**  
 $|\vec{r}'(t)| = |\langle x'(t), y'(t) \rangle|$  **speed**  
 $\vec{r}''(t) = \langle x''(t), y''(t) \rangle$  **acceleration**  
 $\vec{r}'''(t) = \langle x'''(t), y'''(t) \rangle$  **jerk**

### SPACE CURVE

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$  **position**  
 $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$  **velocity**  
 $|\vec{r}'(t)| = |\langle x'(t), y'(t), z'(t) \rangle|$  **speed**  
 $\vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle$  **acceleration**  
 $\vec{r}'''(t) = \langle x'''(t), y'''(t), z'''(t) \rangle$  **jerk**

TANGENT DIRECTION: The velocity vector  $\vec{r}'(t)$  is tangent to the curve at  $\vec{r}(t)$  because  $\vec{r}'(t) \sim \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$ .

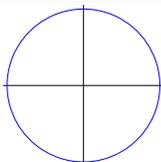
Note that the second derivative  $\vec{r}''(t)$  can point in any direction. It is by Newton's law proportional to the force acting on the body.

ARC LENGTH. If  $t \in [a, b] \mapsto \vec{r}(t)$  with velocity  $\vec{r}'(t)$  and speed  $|\vec{r}'(t)|$ , then  $\int_a^b |\vec{r}'(t)| dt$  is called the **arc length of the curve**. For space curves for example, this can be written out as

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

Note that the arc length is a scalar. The integral  $\int_0^t \vec{r}'(s) ds$  we have seen for example in the free fall is a vector.

CIRCLE. The arc length of the circle  $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$  parameterized by  $0 \leq t \leq 2\pi$  is  $2\pi$  because the speed  $|\vec{r}'(t)|$  is constant and equal to 1.

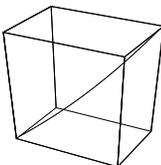


OPPORTUNITY. Compute the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), t^2/2 \rangle.$$

for  $1 \leq t \leq 2$ . Because  $\vec{r}'(t) = \langle 1, 1/t, t \rangle$ , we have  $|\vec{r}'(t)| = \sqrt{1 + \frac{1}{t^2} + t^2} = \frac{1}{t} + t$ .

We have  $L = \int_1^2 \frac{1}{t} + t dt = \log(t) + \frac{t^2}{2} \Big|_1^2 = \log(2) + 2 - 1/2$ . This curve does not have a name. It is constructed in such a way that the arc length can be computed. I name it "opportunity curve".



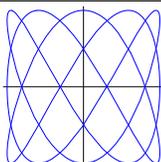
PARAMETER INDEPENDENCE. The arc length is independent of the parameterization of the curve.

PROOF: changing the parameter corresponds to a substitution in the integration.

EXAMPLE. The circle parameterized by  $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$  on  $t = [0, \sqrt{2\pi}]$  has the velocity  $\vec{r}'(t) = 2t(-\sin(t), \cos(t))$  and speed  $2t$ . The arc length is  $\int_0^{\sqrt{2\pi}} 2t dt = t^2 \Big|_0^{\sqrt{2\pi}} = 2\pi$ .

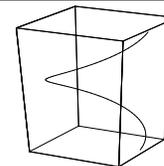
PARAMETERIZATION BY ARC-LENGTH. If  $\sigma(t) = \int_0^t |\vec{r}'(s)| ds$  is the **arc length function**. Then  $\vec{p}(s) = \vec{r}(\sigma^{-1}(s))$  is called the **parameterization by the arc length**. Example: if  $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$ , then  $\sigma(t) = \int_0^t 2\tau d\tau = t^2 = s$  which gives  $t = \sigma^{-1}(s) = \sqrt{s}$ . Therefore  $\vec{p}(s) = \vec{r}(\sqrt{s}) = \langle \cos(s), \sin(s) \rangle$  is the arc length parameterization.

REMARK. Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure**  $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$  has the arc length  $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)} dt$ . This integral must be evaluated numerically. In the Mathematica Lab, you will see how to do that with the computer for the ellipse, where the arc length leads to an "elliptic integral".



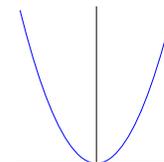
EXAMPLE: LENGTH OF THE HELIX:

The helix  $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$  has velocity  $\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$  and constant speed  $|\vec{r}'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1} = \sqrt{2}$ . The simplicity of this example is the reason that it appears in all calculus text books.



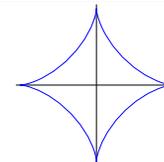
LENGTH OF PARABOLA.

What is the arc length of the curve  $\vec{r}(t) = \langle t, t^2 \rangle$  on  $t \in [-1, 1]$ ? We have  $\int \sqrt{1 + 4t^2} = x\sqrt{1 + 4t^2}/2 + \text{arcsinh}(2t)/4$  using **integration by parts**. The result is  $\sqrt{5} + \text{arcsinh}(2)/2$ . to the integration: integration by parts gives  $I = \int \sqrt{1 + t^2} dt = t\sqrt{1 + t^2} - \int \frac{t^2}{\sqrt{1 + t^2}} dt$  Now add and subtract  $\int \frac{1}{\sqrt{1 + t^2}} dt$  to get  $I = t\sqrt{1 + t^2} - I + \frac{1}{\sqrt{1 + t^2}} dt$  so that  $2I = t\sqrt{1 + t^2} + \text{arcsinh}(t)/2$ .



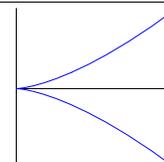
LENGTH OF ASTEROID.

What is the arc length of the curve  $\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$ ? We have  $|\vec{r}'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$ . Therefore,  $\int_0^{2\pi} (3/2)\sin(2t) dt = 6$ .



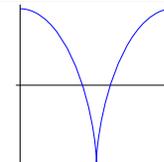
LENGTH OF AN ELLIPTIC CURVE:

What is the arc length of  $\vec{r}(t) = \langle t^2/2, t^3/3 \rangle$  for  $-1 \leq t \leq 1$ . This cubic curve satisfies  $y^2 = x^3/8$  and is an example of an **elliptic curve**. Because  $\int x\sqrt{1 + x^2} dx = (1 + x^2)^{3/2}/3$ , the integral can be evaluated as  $\int_{-1}^1 |x\sqrt{1 + x^2}| dx = 2 \int_0^1 x\sqrt{1 + x^2} dx = 2(1 + x^2)^{3/2}/3 \Big|_0^1 = 2(2\sqrt{2} - 1)/3$ .



LENGTH OF EPICYCLE.

What is the arc length of  $\vec{r}(t) = \langle t + \sin(t), \cos(t) \rangle$  parameterized by  $0 \leq t \leq 2\pi$ . We have  $|\vec{r}'(t)| = \sqrt{2 + 2\cos(t)}$ . so that  $L = \int_0^{2\pi} \sqrt{2 + 2\cos(t)} dt$ . A **substitution**  $t = 2u$  gives  $L = \int_0^\pi \sqrt{2 + 2\cos(2u)} 2du = \int_0^\pi \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)} 2du = 4 \int_0^\pi |\cos(u)| du = 8$ .

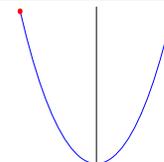


LENGTH OF CATENARY.

Find the arc length of the catenary  $\vec{r}(t) = \langle t, \cosh(t) \rangle$ , where  $\cosh(t) = (e^t + e^{-t})/2$  is the **hyperbolic cosine** and  $t \in [-1, 1]$ . We have

$$\cosh^2(t) - \sinh^2(t) = 1,$$

where  $\sinh(t) = (e^t - e^{-t})/2$  is the **hyperbolic sine**.



CATENARY HISTORY: Galilei was the first to investigate the catenary. It is the curve, a freely hanging heavy rope describes, if the end points have the same height. But Galileo mistook the curve for a parabola. It was Johannes Bernoulli in 1691, who obtained its true form after some competition involving Huygens, Leibniz and two Bernoulli's. The name "catenarian" which means chain curve, was first used by Huygens in a letter to Leibniz in 1690. The picture to the right shows Galileo Galilei

