

**Mathematics 21a Fall 2006**  
**In class problems. Dec. 14**

1. Calculate  $\oint_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = \langle yz + 3e^{x^2}, x(z + 1) + 3e^{y^2}, xy + 3e^{z^2} \rangle$ , and  $C$  is the boundary of the triangle with the vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$  and  $(0, 1, 1)$  oriented counter-clockwise if viewed from above.

2. Calculate the *outward* flux of the vector field  $\vec{F} = \langle 6xy + z, y^2, -6yz - x \rangle$  through the boundary of the unit cube  $\{0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$ .

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1.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle x - x, y - y, z + 1 - z \rangle = \langle 0, 0, 1 \rangle.$$

Using the Stokes' theorem we can replace the contour integral by the surface integral over the given triangle:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dA.$$

Parameterize the triangle by  $(x, y)$ :  $\vec{r} = (x, y, y)$ . The domain of the parameterers in the triangle  $R$  in the  $xy$ -plane.

$$\vec{r}_x = \langle 1, 0, 0 \rangle, \quad \vec{r}_y = \langle 0, 1, 1 \rangle, \quad \vec{r}_x \times \vec{r}_y = \langle 0, -1, 1 \rangle.$$

The normal vector  $\vec{r}_x \times \vec{r}_y$  has the  $z$ -component positive, hence it gives the right orientation.

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dA &= \iint_R \operatorname{curl} \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) \, dx \, dy \\ &= \iint_R \langle 0, 0, 1 \rangle \cdot \langle 0, -1, 1 \rangle \, dx \, dy = \iint_R 1 \, dx \, dy \\ &= \text{Area of } R = \frac{1}{2}. \end{aligned}$$

2.

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial}{\partial x}(6xy + z) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(-6yz - x) \\ &= 6y + 2y - 6y = 2y. \end{aligned}$$

Using the Divergence theorem we can replace the surface integral by the triple integral over the unit cube:

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} \, dA &= \iiint_V 2y \, dV \\ &= \int_0^1 \int_0^1 \int_0^1 2y \, dx \, dy \, dz = 1. \end{aligned}$$