Mathematics 21a Fall 2006
In class problems. Dec. 12

1. Calculate the area of the part of the sphere $x^2 + y^2 + z^2 = 25$ lying above the plane $z = 4$.

2. Calculate the flux $\iint_S \overrightarrow{F} \cdot \vec{n}\, dS$ of the vector field $\overrightarrow{F} = 2x\vec{i} + 4z\vec{k}$ through the part of the paraboloid $z = 4 - x^2 - y^2$ lying in the positive octant. The orientation of the paraboloid is given by a normal vector pointing upward.

1. Let (x, y) be the parameterization of the given part of the sphere. The domain R of the parameters is the disk of radius 3 in the xy-plane. Then

$$\vec{r} = (x, y, \sqrt{25 - x^2 - y^2}), \quad \vec{r}_x = \langle 1, 0, -\frac{x}{\sqrt{25 - x^2 - y^2}} \rangle, \quad \vec{r}_y = \langle 0, 1, -\frac{y}{\sqrt{25 - x^2 - y^2}} \rangle,$$

$$\vec{r}_x \times \vec{r}_y = \langle -\frac{x}{\sqrt{25 - x^2 - y^2}}, -\frac{y}{\sqrt{25 - x^2 - y^2}}, 1 \rangle.$$

The area element is

$$dS = |\vec{r}_x \times \vec{r}_y| \; dx\,dy = \sqrt{\frac{x^2 + y^2 + 25 - x^2 - y^2}{25 - x^2 - y^2}} \; dx\,dy = \frac{5}{\sqrt{25 - x^2 - y^2}} \; dx\,dy.$$

The total area is

$$\iint_S 1\,dS = \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}} \; dx\,dy = \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} r dr\,d\theta$$

$$= 5 \cdot 2\pi \int_0^3 \frac{1}{\sqrt{25 - r^2}} r\,dr = 10\pi \int_{25}^{16} -\frac{1}{2\sqrt{u}} \; du = 10\pi \; \sqrt{u}\Big|_{16}^{25} = 10\pi.$$

2. Parameterize the paraboloid by (x, y) in the quarter disk R of radius 2 in the xy-plane.

$$\vec{r} = (x, y, 4 - x^2 - y^2), \quad \vec{r}_x = \langle 1, 0, -2x \rangle, \quad \vec{r}_y = \langle 0, 1, -2y \rangle,$$

$$\vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1 \rangle.$$

The normal vector $\vec{r}_x \times \vec{r}_y$ has the z-component positive, hence it gives the right orientation.

$$\iint_S \overrightarrow{F} \cdot \vec{n} \; dA = \iint_R \overrightarrow{F} \cdot (\vec{r}_x \times \vec{r}_y) \; dx\,dy = \iint_R \langle 2x, 0, 4(4 - x^2 - y^2) \rangle \cdot \langle 2x, 2y, 1 \rangle \; dx\,dy$$

$$= \iint_R (16 - 4y^2) \; dx\,dy = \int_0^{\pi/2} \int_0^2 (16 - 4r^2 \sin^2\theta) \; r\,dr\,d\theta = \int_0^{\pi/2} (8r^2 - r^4 \sin^2\theta)\Big|_{r=0}^2 \; d\theta$$

$$= \int_0^{\pi/2} (32 - 16\sin^2\theta) \; d\theta = \int_0^{\pi/2} (32 - 16\frac{1 - \cos 2\theta}{2}) \; d\theta = \int_0^{\pi/2} (24 + 8\cos 2\theta) \; d\theta$$

$$= 24\frac{\pi}{2} + 4\sin 2\theta\Big|_0^{\pi/2} = 12\pi.$$