Mathematics 21a Fall 2006
In class problems. Nov. 28

1. Let T be the solid in the *positive* octant that is *inside* the sphere $x^2 + y^2 + z^2 = 4$ and *outside* the cone $z = \sqrt{x^2 + y^2}$. Assume that the density of the solid is given by $\rho(x, y, z) = z(x^2 + y^2)$. Find its total mass.

2. Find the moment of intertia around the x-axis of the solid with density $\rho = 1$ that lies inside both the sphere $x^2 + y^2 + z^2 = 2$ and the cylinder $y^2 + z^2 = 1$.

1. In spherical cordinates the sphere is $r = 2$, the cone is $\phi = \pi/4$ and the density function is $\rho = r \cos \phi \cdot r^2 \sin^2 \phi$. The first octant is defined by $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi/2$. The total mass in these coordinates will be

$$M = \iiint_V \rho \, dV = \int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^2 r^3 \cos \phi \sin^2 \phi \cdot r^2 \sin \phi \, dr \, d\phi \, d\theta$$

$$= \int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^2 r^5 \sin^3 \phi \cos \phi \, dr \, d\phi \, d\theta = \frac{\pi}{2} \int_{\pi/4}^{\pi/2} \frac{r^6}{6} \Big|_0^2 \sin^3 \phi \cos \phi \, d\phi$$

$$= \frac{64\pi}{12} \frac{\sin^4 \phi}{4} \Big|_{\pi/4}^{\pi/2} = \frac{4\pi}{3} \left(1 - \left(\frac{1}{\sqrt{2}} \right)^4 \right) = \pi.$$

2. We will use the cylindrical coordinates $x = x$, $y = r \cos \theta$, $z = r \sin \theta$.

$$I_x = \iiint_V (y^2 + z^2) \rho \, dV =$$

$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r^2 r \, dx \, dr \, d\theta = 2\pi \int_0^1 2\sqrt{2-r^2} \, r^3 \, dr$$

$$= 2\pi \int_2^1 -\sqrt{u}(2-u) \, du, \qquad \text{where } u = 2 - r^2, \ du = -2r \, dr$$

$$= 2\pi \int_1^2 \sqrt{u}(2-u) \, du = 2\pi \int_1^2 2u^{\frac{1}{2}} - u^{\frac{3}{2}} \, du$$

$$= 2\pi \left(\frac{4}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right) \Big|_1^2 = 2\pi \left(\frac{4}{3} 2\sqrt{2} - \frac{2}{5} 4\sqrt{2} - \frac{4}{3} + \frac{2}{5} \right) = \frac{4\pi}{15} (8\sqrt{2} - 7).$$