Mathematics 21a Fall 2006
In class problems. Nov. 21

1. Evaluate $\iiint_V xy\,dV$, where V is the positive eighth of the unit ball.

2. Find the center of mass of a constant density pyramid with square base of side $\sqrt{2}$ and height 1. Hint: Position the pyramid in $\mathbb{R}^3$ such that its five vertices are $(\pm 1, 0, 0), (0 \pm 1, 0), (0, 0, 1)$.

3. (Bonus) Can you set up the interated integral over the domain which is a square if looked at from the x-axis, a trangle from the y-axis and a disk from the z-axis?

1. We replace $\iiint\limits_V xy\,dV$ *by the iterated integral*

$$\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} xy\,dx\,dy\,dz = \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{2}y(1-y^2-z^2)\,dy\,dz$$

$$= \int_0^1 \frac{1}{4}y^2 - \frac{1}{8}y^4 - \frac{1}{4}y^2z^2 \Big|_0^{\sqrt{1-z^2}} dz = \int_0^1 \frac{1}{4} - \frac{1}{4}z^2 - \frac{1}{8} + \frac{1}{4}z^2 - \frac{1}{8}z^4 - \frac{1}{4}z^2 + \frac{1}{4}z^4\,dz = \frac{1}{15}$$

2. Let ρ_0 *be the density. By symmetry* $\bar{x} = \bar{y} = 0$*. To calculate* $\bar{z}$ *we break the pyramid into 4 tetrahedrons.*

$$\bar{z} = \frac{\iiint\limits_V z\rho_0\,dV}{\iiint\limits_V \rho_0\,dV} = \frac{4\int_0^1 \int_0^{1-z} \int_0^{1-y-z} \rho_0 z dx\,dy\,dz}{4\int_0^1 \int_0^{1-z} \int_0^{1-y-z} \rho_0 dx\,dy\,dz} = \frac{\int_0^1 \int_0^{1-z} (1-y-z)z\,dy\,dz}{\int_0^1 \int_0^{1-z} (1-y-z)z\,dy\,dz}$$

$$= \frac{\int_0^1 z(1-z)(1-z) - z\frac{(1-z)^2}{2}\ dz}{\int_0^1 (1-z)(1-z) - \frac{(1-z)^2}{2}\ dz} = \frac{\int_0^1 z\frac{(1-z)^2}{2}\ dz}{\int_0^1 \frac{(1-z)^2}{2}\ dz} = \frac{\int_0^1 z - 2z^2 + z^3\ dz}{\int_0^1 1 - 2z + z^2\ dz} = \frac{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}}{1 - 1 + \frac{1}{3}} = \frac{1}{4}.$$