GREEN'S THEOREM — Math21a, O. Knill

HOMEWORK: 15.4: 8,20,28,34,42

LINE INTEGRALS. If $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ is a vector field and $C : \vec{r}(t) = \langle x(t), y(t)\rangle, t \in [a,b]$ is a curve, we defined the **line integral**

$$\int_C \vec{F}\ d\vec{r} = \int_a^b \vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t)\ dt$$

THE CURL OF A 2D VECTOR FIELD. The **curl** of a 2D vector field $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ is defined as the scalar field

$$\text{curl}(F)(x,y) = N_x(x,y) - M_y(x,y)\ .$$

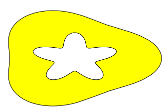
Our book does not introduce the curl in two dimensions. We do. We will see in the end of the course why.

INTERPRETATION. curl(F) measures the **vorticity** of the vector field. One can write $\nabla \times \vec{F} = \text{curl}(F)$ for the curl of F because the 2D cross product of (∂_x, ∂_y) with $\vec{F} = \langle M, N\rangle$ is $N_x - M_y$.

EXAMPLES.
1) For $\vec{F}(x,y) = \langle -y, x\rangle$ we have $\text{curl}(F)(x,y) = 2$.
2) If $\vec{F}(x,y) = \nabla f$, (conservative field = gradient field = potential) then $M(x,y) = f_x(x,y), N(x,y) = f_y(x,y)$ and $\text{curl}(F) = N_x - M_y = f_{yx} - f_{xy} = 0$.

GREEN'S THEOREM. (1827) If $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ is a vector field and G is a region for which the boundary is a curve parametrized so that G is "to the left". Then

$$\int_C F \cdot ds = \int\int_G \text{curl}(F)\ dxdy$$



Note that for a region with holes, the boundary consists of many curves. They are always oriented so that G **is to the left**. Greens theorem is a theorem in two dimensions.

GEORGE GREEN (1793-1841) was one of the most remarkable physicists of the nineteenth century. He was a self-taught mathematician and miller, whose work has contributed greatly to modern physics.

SPECIAL CASE. If $\vec{F}$ is a gradient field, then both sides of Green's theorem are zero:

$\int_C \vec{F} \cdot d\vec{r}$ is zero by the fundamental theorem for line integrals.	$\int\int_G \text{curl}(F) \cdot dA$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(f)) = 0$.

$\text{curl}(\text{grad}(f)) = 0$ can be checked directly by writing it as $\nabla \times \nabla f$ and using that the cross product of two identical vectors is 0. Just treat ∇ as a vector.

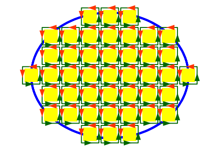
APPLICATION: CALCULATING LINE INTEGRALS. Sometimes, the calculation of line integrals is harder than calculating a double integral. Example: find the line integral of $\vec{F}(x,y) = \langle x^2 - y^2, 2xy\rangle = \langle M, N\rangle$ along the boundary of the rectangle $[0,2] \times [0,1]$. Solution: $\text{curl}(F) = N_x - M_y = 2y - 2y = -4y$ so that $\int_C F\ dr = \int_0^2 \int_0^1 4y\ dydx = 2y^2|_0^1 x|_0^2 = 4$.

Remark. One can easily find examples, where a direct calculation of the line integral is not possible in closed form, but where Green allows to do it.

WHERE IS THE PROOF? (Quote: General Hein in "Final Fantasy").

Look first at a small square $G = [x, x+\epsilon] \times [y, y+\epsilon]$. The line integral of $\vec{F} = \langle M, N\rangle$ along the boundary is $\int_0^\epsilon M(x+t, y)dt + \int_0^\epsilon N(x+\epsilon, y+t)\ dt - \int_0^\epsilon M(x+t, y+\epsilon)\ dt - \int_0^\epsilon N(x, y+t)\ dt$. (Note also that this line integral measures the "circulation" at the place (x,y).)

Because $N(x+\epsilon, y) - N(x,y) \sim N_x(x,y)\epsilon$ and $M(x, y+\epsilon) - M(x,y) \sim M_y(x,y)\epsilon$, the line integral is $(N_x - M_y)\epsilon^2$ is about the same as $\int_0^\epsilon \int_0^\epsilon \text{curl}(F)\ dxdy$. All identities hold in the limit $\epsilon \to 0$.
To prove the statement for a general region G, we chop it into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the rectangles is a Riemann sum approximation of the double integral.



APPLICATION: CALCULATING DOUBLE INTEGRALS.

Sometimes, it is harder to calculate the double integral. An example is to determine the area of a polygon with sides $(x_1, y_1), ..., (x_n, y_n)$ In that case, the vector field $\vec{F} = (-y, x)/2$ leads to the closed formula for the area: $A = \frac{1}{2}\sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i)$, something a computer can do evaluate very fast and which does not involve any integration.

APPLICATION: FINDING THE CENTROID OF A REGION.
Green's theorem allows to express the coordinates of the centroid

$$\left(\int\int_G x\ dA/A, \int\int_G y\ dA/A\right)$$

as line integrals.

For example, to verify

$$\int\int_G x\ dA = \int_C F\ dr$$

take the vector field $F\ dr = x^2 dy$.

APPLICATION: AREA FORMULAS.

The vector fields $\vec{F}(x,y) = \langle M, N\rangle = \langle -y, 0\rangle$ or $\vec{F}(x,y) = (0, x)$ have vorticity $\text{curl}(F(x,y)) = 1$. The right hand side in Green's theorem is the **area** of G:

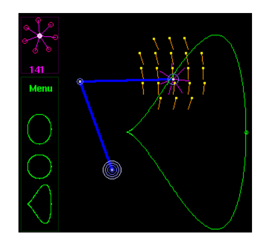
$$\text{Area}(G) = \int_C -y\ dx = \int_C x\ dy$$

EXAMPLE. Let G be the region under the graph of a function $f(x)$ on $[a,b]$. The line integral around the boundary of G is 0 from $(a,0)$ to $(b,0)$ because $F(x,y) = 0$ there. The line integral is also zero from $(b,0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a,0)$ because $N = 0$. The line integral on $(t, f(t))$ is $-\int_a^b ((-y(t), 0) \cdot (1, f'(t))dt = \int_a^b f(t)\ dt$. Green's theorem assures that this is the area of the region below the graph.

APPLICATION. $\text{curl}(F) = 0$ in a simply connected region implies F is conservative:
Proof: Given a closed curve C. It encloses a region G. Greens theorem assures that $\int_C F\ dr = 0$. So F has the closed loop property and is therefore a gradient field.

APPLICATION. THE PLANIMETER.

The planimeter is a mechanical device for measuring areas: in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles. There is a vector field F associated to a planimeter (put a vector of length 1 orthogonally to the arm). One can prove that F has vorticity 1. The planimeter calculates the line integral of F along a given curve. Green's theorem assures it is the area.



The picture to the right shows a Java applet which allows to explore the planimeter (from a CCP module by O. Knill and D. Winter, 2001).
To explore the planimeter, visit the URL http://www.math.duke.edu/education/ccp/materials/mvcalc/green/