THE GRADIENT **Math21a**

Section 13.6: 2,8 Section: 13.7: 40, 42
Problem A: given $g(x,y,z) = z - f(x,y)$. What is the relation between the gradient of g, a vector in space and the gradient of f, a vector in the plane? Describe it in general and explain it also with a simple example, where you can draw the level curves of f and the level surfaces of g.

NABLA. The symbol ∇ is called "Nabla".

GRADIENT. Define the **gradient** of a function $f(x,y)$ as

$$\nabla f(x,y) = (f_x(x,y), f_y(x,y))$$

or

$$\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$$

if the function has three variables.

ETYMOLOGY. The name "Nabla" means "Egyption harp". Its origin is believed to be related to the Hebrew word "nevel"=harp which seems to have the same aramaic origin.

REMINDER CHAIN RULE. The chain rule in multivariable calculus can be written more compactly as

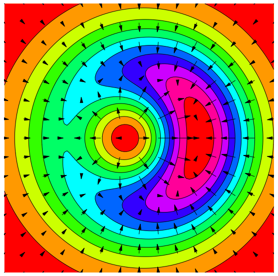
$$\boxed{\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$$

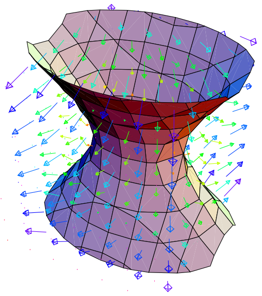
It looks like the one-dimensional chain rule, but the derivative f' is replaced with the gradient and the derivative of $\vec{r}$ is the velocity $\vec{r}\,'(t)$.

GRADIENTS AND LEVEL CURVES/SURFACES.

Gradients are orthogonal to level curves and level surfaces.

PROOF. Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}\,'(t)$.

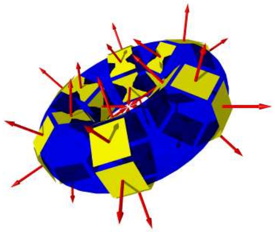




REMINDER TANGENT LINE AND TANGENT PLANE. Because $\vec{n} = \nabla f(x_0,y_0) = \langle a,b \rangle$ is perpendicular to the level curve $f(x,y) = c$ through (x_0,y_0), the equation for the tangent line is $ax + by = d$, $a = f_x(x_0,y_0)$, $b = f_y(x_0,y_0)$, $d = ax_0 + by_0$. This can be written ccompactly as

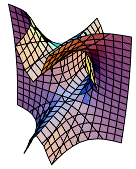
$$\boxed{\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0}$$

This equation means that the gradient of f is perpendicular to any vector $(\vec{x} - \vec{x}_0)$ in the plane.



PROBLEM. Compute the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1,1,1)$.

SOLUTION: $\nabla f(x,y,z) = \langle 6xy, 3x^2, 2z \rangle$. And $\nabla f(1,1,1) = \langle 6,3,2 \rangle$. The plane is $6x + 3y + 2z = 11$.



REMINDER: LINEARIZATION. Using the gradient, we can write the linearization of $f(x,y)$

$$\boxed{L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)}$$

This is the analogue of $L(x) = f(x_0) + f'(x_0)(x - x_0)$ in one dimensions.

PROBLEM. Reflect the ray $\vec{r}(t) = \langle 1-t, -t, 1 \rangle$ at the surface

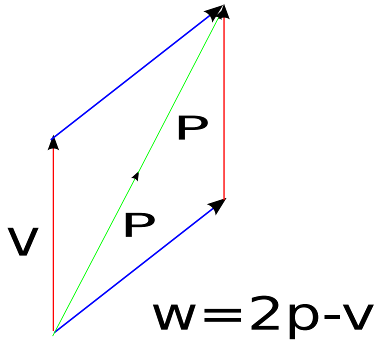
$$x^4 + y^2 + z^6 = 6 .$$

Solution. $\vec{r}(t)$ hits the surface at the time $t = 2$ in the point $(-1,-2,1)$. The velocity vector in that ray is $\vec{v} = \langle -1,-1,0 \rangle$ The normal vector at this point is $\nabla f(-1,-2,1) = \langle -4,4,6 \rangle = \vec{n}$.

Remember the formula for the reflection derived in the midterm? It was

$$R(\vec{v} = 2\mathrm{Proj}_{\vec{n}}(\vec{v}) - \vec{v} .$$

We have $\mathrm{Proj}_{\vec{n}}(\vec{v}) = 8/68\langle -4,-4,6 \rangle$. Therefore, the reflected ray is $\vec{w} = (4/17)\langle -4,-4,6 \rangle - \langle -1,-1,0 \rangle$.



THE GRADIENT IN OPTICS.

The **Eiconal equation** in optics states $||\nabla f|| = 1$. The level curves of the solution f are wave fronts.