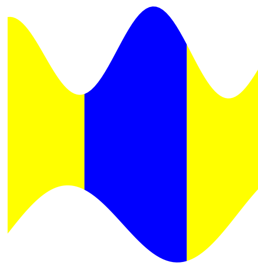
2D INTEGRALS FOR SIMPLE REGIONS — Math21a, O. Knill

Y-SIMPLE REGIONS. A class of regions is what is bound between the graphs of two functions $c(x)$ and $d(x)$. Such regions are sometimes called y-**simple regions**. One can write the region as

$$R = \{(x,y) \mid c(x) \le y \le d(x)\ \} .$$

An integral over such a region is an iterated integral which is:

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx$$

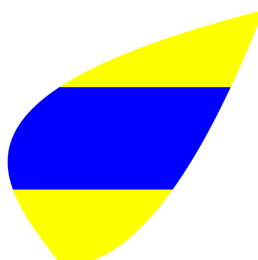


X-SIMPLE REGIONS. It is defined by two functions $a(y)$ and $b(y)$ which are functions of y. One can write the region as

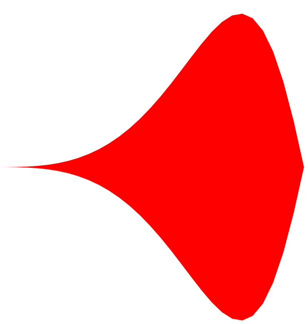
$$R = \{(x,y) \mid a(y) \le x \le b(y)\ \} .$$

An integral over such a region is an iterated integral:

$$\iint_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy$$



EXAMPLE 1) Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the grapy of $-\sin(x^3)$ for $0 \le x \le \pi$. The value of this integral has a physical meaning. It is a moment of inertia. We will come back to that next week.

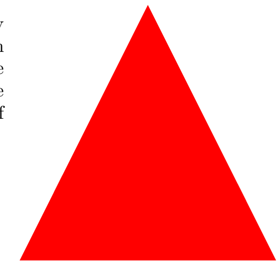


$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\ dydx = 2 \int_0^{\pi^{1/3}} \sin(x^3) x^2\ dx =$$

Now, we have an integral, which we can solve by substitution

$$= -(2/3)\cos(x^3)|_0^{\pi^{1/3}} = 4/3$$

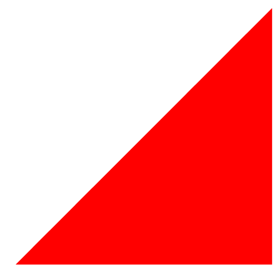
EXAMPLE 2) Integrate $f(x,y) = y^2$ over the region bound by the $x-$ axes, the lines $y = x+1$ and $y = 1-x$. The problem is best solved as a x-simple integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of y: they are $x = y-1$ and $x = 1-y$



$$\int_0^1 \int_{y-1}^{1-y} y^3\ dx\ dy = 2\int_0^1 y^3(1-y)\ dy = 2(1/4 - 1/3) = 1/10 .$$

EXAMPLE 3). Let R be the triangle $1 \ge x \ge 0, 0 \le y \le x$. What is

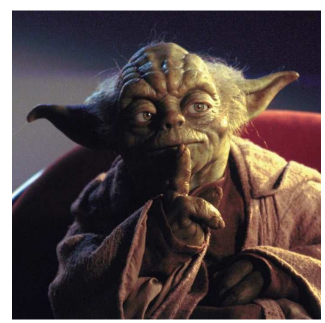
$$\int\int_R e^{-x^2}\ dxdy\ ?$$



The x-simple integral $\int_0^1 [\int_y^1 e^{-x^2}\ dx] dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions.

The y-simple integral $\int_0^1 [\int_0^x e^{-x^2}\ dy]\ dx$ however can be solved:

$$= \int_0^1 x e^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316... .$$



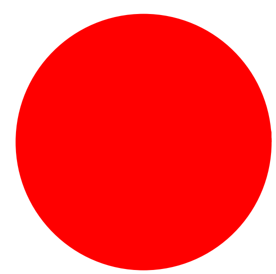
WORDS OF WISDOM:

> If a double integral you can not solve, the order of integration change you must.

> For solving double integrals, a picture at hand must be.

EXAMPLE 4: THE AREA OF A DISC OF RADIUS R:

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1\ dydx = \int_{-R}^{R} \sqrt{R^2-x^2}\ dx .$$



This integral can be solved with the substitution $x = R\sin(u), dx = R\cos(u)$

$$\int_{-\pi/2}^{\pi/2} \sqrt{R^2 - R^2\sin^2(u)} R\cos(u)\ du = \int_{-\pi/2}^{\pi/2} R^2\cos^2(u)\ du$$

Now continue with a trigonometc identity to get $R^2 \int_{-\pi/2}^{\pi/2} \frac{(1+\cos(2u))}{2}\ du = R^2\pi$. This is too complicated. We will see how to do that better in polar coordinates.