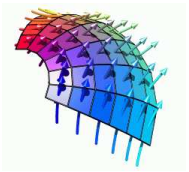
FLUX INTEGRALS — Math21a, O. Knill

HOMEWORK: 15.6: 16, Problem A: compute the surface area of the surface $\vec{r}(u,v) = \langle v\cos(u), v\sin(u), v^2\rangle$, with $0 \leq u \leq \pi/2$ and $0 \leq v \leq 1$. 15.7: 28, 30 (compute only the flux integral), Problem B: compute the flux of $\vec{F}(x,y,z) = \langle 0,0,z\rangle$ through the torus parameterized as $\vec{r}(u,v) = \langle (2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v)\rangle$, where both u and v range from 0 to $2pi$.

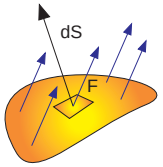
FLUX INTEGRAL OF A VECTOR FIELD THROUGH SURFACE. If S is a surface, and $\vec{F}$ is a vector field in space we define an integral $\int\int_S \vec{F}\cdot d\vec{S}$ which is called the **flux** of $\vec{F}$ through the surface S.



DEFINITION. Given a surface $S = \vec{r}(G)$, where G is a domain in the plane and where $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ is the parameterization of the surface. The **flux integral** of F through S is defined as the double integral

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_G \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\, dudv$$

The notation $d\vec{S} = (\vec{r}_u \times \vec{r}_v)\, dudv$ represents an infinitesimal vector in the normal direction to the surface.



INTERPRETATION. If $\vec{F}$ = fluid velocity field, then $\int\int_S \vec{F}\cdot d\vec{S}$ is the flux of fluid passing through S. If $\vec{F}$ is a fluid, then the integral gives the amount of fluid which passes through S in a unit time.

EXAMPLE. Let $F(x,y,z) = (0,1,z^2)$ and let S be the sphere with

$$\vec{r}(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$$

, $\vec{r}_u \times \vec{r}_v = \sin(v)\vec{r}$ so that $F(\vec{r}(u,v)) = (0,1,\cos^2(v))$ and

$$\int_0^{2\pi}\int_0^{\pi} \sin(v)(0,1,\cos^2(v))\cdot(\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))\, dudv\ .$$

The integral is $\int_0^{2\pi}\int_0^{\pi} \sin^2(v)\sin(u) + \cos^3(v)\sin(v)\, dudv = 0$.
Most of the flux of this vector field comes from the south pole, most flux passes out at the north pole. However, there is some symmetry and what enters in the south leaves in the north. The total flux is zero.

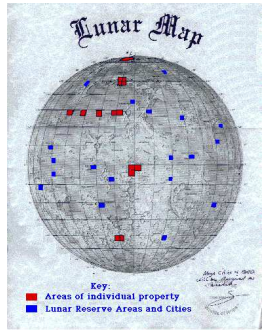
WHAT IS THE FLUX INTEGRAL? Because $n = \vec{r}_u \times \vec{r}_v/|\vec{r}_u \times \vec{r}_v|$ is a unit vector normal to the surface and on the surface, $\vec{F}\cdot\vec{n}$ is the normal component of the vector field with respect to the surface, we can write $\int\int_S \vec{F}\cdot d\vec{S} = \int\int \vec{F}\cdot\vec{n}\, dS$. The function $\vec{F}\cdot\vec{n}$ is the scalar projection of $\vec{F}$ in the normal direction. Whereas the formula $\int\int 1\, dS$ gave the area of the surface with $dS = |r_u \times r_v|dudv$, the flux integral weights each area element dS with the normal component of the vector field with $(F(r(u,v))\cdot n(r(u,v))$.

BIKING IN THE RAIN.
Do you also experience that in the rain you get soaked more when biking or walking fast then by biking or walking slowly. Why? For a biker who drives with speed V along the x axis, the rain is a fluid with velocity $\vec{F} = \langle -V,0,W\rangle$, where W is the speed of the rain drops falling along the z-axis. If the biker is a rectangular box $[0,1/2]\times[0,1]\times[1/4,1/3]$ we can calculate the flux through that surface. Parameterize the front by $\vec{r}(u,v) = \langle 0,u,v\rangle$ in the yz-plane. The water soaked up in unit time by the front clothes is the flux of the rain through that surface. We know $\vec{r}_u \times \vec{r}_v = \langle 1,0,0\rangle$ and $\int\int_S \vec{F}\cdot d\vec{S} = \int_0^{1/2}\int_0^1 \langle -V,0,W\rangle\cdot\langle 1,0,0\rangle\, dudv = -V/2$. The flux through the side surface is zero (if the rain falls vertically) and the flux through the top is constant, namely $W/12$. The total flux is $W/12 - V/2$. You catch more rain if you drive faster.

PROPERTY ON THE MOON. A few years ago, in lack of a better present idea, I bought property on the moon for family (For 30 Us-Dollars one could get 1'777.58 Acres)) at the Lunar Embassy (www.moonshop.com). An acre is now sold for 19.99 Dollars plus 1.51 Dollars lunar tax. You can also buy an entire "city" for 4500 dollars now.

The coordinates of the parcel are $34E, 26N$. The moon has a radius of $\rho = 1737km$. The surface is $3.810^6 km^2$. Multiply this by 247 gives about 10 billion (10^{10}) acres (more than an acre for each person on the world). The moon surface is parameterized by $\vec{r}(u,v) = \rho\langle\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle$, where u is the latitude and $\pi/2 - v$ is the latitude. If we look at the moon from the direction $(1,0,0)$, how big does the area of a real estate $S = \vec{r}([a,b]\times[c,d])$ appear?

Lunar Deed



The flux of the vector field $(1,0,0)$ through the surface is $\int\int_S \vec{F}\cdot d\vec{S} = \int_a^b\int_c^d r^2\cos(u)\sin^2(v)dvdu$.
This is also the area we **see**. You notice that the visible area is small for $\cos(u)$ small or $\sin(v)$ small, which is near $90E, 90W$ or near the poles.

EXAMPLE. Calculate the flux of the vector field $\vec{F}(x,y,z) = \langle 1,2,4z\rangle$ through the paraboloid $z = x^2+y^2$ lying above the region $x^2+y^2 \leq 1$.

We can parametrize with polar coordinates $\vec{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r^2\rangle$ where $\vec{r}_r \times \vec{r}_\theta = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r\rangle$ and $\vec{F}(\vec{r}(u,v)) = \langle 1,2,4r^2\rangle$. We get $\int_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 (-2r^2\cos(v) - 4r^2\sin(v) + 4r^3)\, drd\theta = 2\pi$.

WHERE CAN FLUX INTEGRALS APPEAR?

- **Fluid dynamics** The flux of the velocity vector field of a membrane through a surface is the volume of the fluid which passes through that surface in unit time. Often one looks at incompressible fluids which means $\mathrm{div}(\vec{v}) = 0$. In this case, the amount of fluid which passes through a closed surface like a sphere is zero because what enters also has toe leave the surface.
- **Electromagnetism** The change of the flux of the magnetic field through a surface is related to the voltage at the boundary of the surface. We will see why one of the Maxwell equations $\mathrm{div}(B) = 0$ means that there are no **magnetic monopoles** and see how divergence is related to flux.
- **Magnetohydrodynamics** Magnetohydrodynamics deals with electrically charged "fluids" like plasmas in the sun or solar winds etc. The velocity field of the wind induces a current and the flux of the field trough a surface can be used to calculate the magnetic field.
- **Gravity** We will see that the flux of a field through a closed surface like the sphere is related to the amount of source inside the the surface. In gravity, where $\vec{F}$ is the gravitational field, the flux of the gravitational field through a closed surface is proportional to the mass inside the surface. Mass is a "source for gravitational field".
- **Thermodynamics**. The heat flux through a surface like the walls of a house are related to the amount of heat which is produced inside the house. A badly isolated house needs has a large heat flux and needs a large source of heating.
- **Particle physics**. When studying the collision of particles, one is interested in the relation between the incoming flux and out-coming fluxes depending on the direction. The key word is: "differential cross-section".
- **Optics**. Measurements of "light fluxes" are important for lasers, astronomy, photography.

PREVIEW FOR NEXT TIME. We can often simplify the computation of the flux by a formula which we cover next time. It is called Stokes theorem.