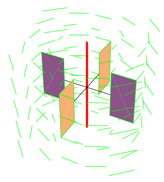
CURL AND DIV — Math21a, O. Knill

HOMEWORK: Section 15.4: 8,20,28,34,42

CURL (3D). The curl of a 3D vector field $\vec{F} = \langle M,N,R\rangle$ is defined as the 3D vector field

$$\text{curl}\langle M,N,R\rangle = \langle R_y - N_z, M_z - R_x, N_x - M_y\rangle .$$

CURL (2D). The curl of a 2D vector field $\vec{F} = \langle M,N\rangle$ is $N_x - M_y$, a scalar field. (In n dimensions, the curl is an object with the dimension $n(n-1)/2$).



PADDLE WHEEL: The wheel indicates the curl vector if $\vec{F}$ is thought of as a wind velocity field. As we will see later the direction in which the wheel turns fastest, is the direction of curl($\vec{F}$). The wheel could actually be used to measure the curl of the vector field at each point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl.

DIV (3D). The **divergence** of $\vec{F} = \langle M,N,R\rangle$ is the scalar field div$\langle M,N,R\rangle = \nabla\cdot F = M_x + N_y + R_z$. DIV (2D). The **divergence** of a vector field $\vec{F} = \langle M,N\rangle$ is div$(M,N) = \nabla\cdot F = M_x + N_y$.

NABLA CALCULUS. With the "vector" $\nabla = (\partial_x, \partial_y, \partial_z)$, we can write curl($\vec{F}$) $= \nabla\times\vec{F}$ and div($\vec{F}$) $= \nabla\cdot\vec{F}$. This is both true in 2D and 3D.

LAPLACE OPERATOR. $\Delta f = \text{divgrad}(f) = f_{xx} + f_{yy} + f_{zz}$ can be written as $\nabla^2 f$ because $\nabla\cdot(\nabla f) = \text{div}(\text{grad}(f)$. One can extend this to vectorfields $\Delta\vec{F} = (\Delta M, \Delta N, \Delta R)$ and write $\nabla^2\vec{F}$.

IDENTITIES. While direct computations can verify the identities to the left, they become evident with Nabla calculus from formulas for vectors like $\vec{v}\times\vec{v} = \vec{0}, \vec{v}\cdot\vec{v}\times\vec{w} = 0$ or $u\times(v\times w) = v(u\cdot w) - (u\cdot v)w$.

$\text{div}(\text{curl}(\vec{F})) = 0$.	$\nabla\cdot\nabla\times\vec{F} = 0$.
$\text{curlgrad}(\vec{F}) = \vec{0}$	$\nabla\times\nabla\vec{F} = \vec{0}$.
$\text{curl}(\text{curl}(\vec{F})) = \text{grad}(\text{div}(\vec{F}) - \Delta(\vec{F}))$.	$\nabla\times\nabla\times\vec{F} = \nabla(\nabla\cdot\vec{F}) - (\nabla\cdot\nabla)\vec{F}$.

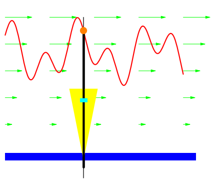
QUIZZ. Is there a vector field $\vec{G}$ such that $\vec{F} = \langle x+y, z, y^2\rangle = \text{curl}(\vec{G})$? Answer: no because div($\vec{F}$) = 1 is incompatible with div(curl($\vec{G}$)) = 0.

ADDENDA TO GREEN'S THEOREM. Green's theorem, one of the advanced topics in this course is useful in physics. We have already seen the following applications

- Simplify computation of double integrals
- Simplify computation of line integrals
- Formulas for centroid of region
- Formulas for area
- Justifying, why mechanical integrators like the planimeter works.

THE CONE PLANIMETER.

The **cone planimeter** is a mechanical instrument to find the antiderivative of a function $f(x)$. It uses the fact that the vector field $F(x,y) = (y,0)$ has the curl$(F) = -1$. By Greens theorem, the line integral around the type I region bounded by 0 and the graph of $f(x)$ in the counter clockwise direction is $\int_a^x f(x)\,dx$. The planimeter determines that line integral.



APPLICATION THERMODYNAMICS. Gases or liquids are often described in a $P-V$ **diagram**, where the volume in the x-axis and the pressure in the y axis. A periodic process like the pump in a refrigerator leads to closed curve $\gamma : r(t) = (V(t), p(t))$ in the $V-p$ plane. The curve is parameterized by the time t. At a given time, the gas has volume $V(t)$ and a pressure $p(t)$. Consider the vector field $F(V,p) = (p,0)$ and a closed curve γ. What is $\int_\gamma F\ ds$? Writing it out, we get $\int_0^t (p(t),0)\cdot(V'(t),p'(t)\ dt = \int_0^t p(t)V'(t)dt = \int_0^t pdV$. The curl of $F(V,p)$ is -1. You see by Green's theorem the integral $-\int_0^t pdV$ is the area of the region enclosed by the curve.

APPLICATION ELECTROMAGNETISM. You look at the Maxwell equations in the homework, where the current $j = 0$ and charges $\rho = 0$. Here c = speed of light.

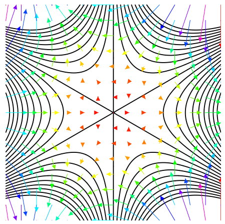
$\text{div}(B) = 0$	No monopoles	there are no magnetic monopoles.
$\text{curl}(E) = -\frac{1}{c}B_t$	Faraday's law	change of magnetic flux induces voltage
$\text{curl}(B) = \frac{1}{c}E_t + \frac{4\pi}{c}j$	Ampère's law	current or change of E produces magnetic field
$\text{div}(E) = 4\pi\rho$	Gauss law	electric charges are sources for electric field

APPLICATION FLUID DYNAMICS. In fluid dynamics, we are interested in the velocity field $F = v$, the density ρ and the potential function f if it exists.

Incompressibility	$\text{div}(v) = 0$	incompressible fluids, in 2D: $v = \text{grad}(u)$
Irrotational	$\text{curl}(v) = 0$	irrotation fluids
Incompressible Irrotational	$\Delta(f) = 0, v = \nabla f$	incompressible, irrotational fluids
Continuity equation	$\dot{\rho} + \text{div}(\rho v) = 0$	no fluid gets lost

An example of a field, which is both incompressible and irrotational is obtained by taking a function f which satisfies $\Delta f = 0$, like $f(x,y) = x^3 - 3xy^2$, then define $F = \nabla f$. In that case, this gives

$$F(x,y) = \langle 3x^2 - 3y^2, -6xy\rangle .$$



AN OTHER VERSION OF GREENS THEOREM. If we rotate the vector field $F = (M,N)$ by 90 degrees $= \pi/2$, we get a new vector field $G = (-N, M)$. The integral $\int_\gamma F\ \cdot ds$ becomes a **flux** $\int_\gamma G\cdot dn$ of G through the boundary of R, where dn is a normal vector with the length of $dr = |r'|dt$. With div$(F) = (M_x + N_y)$, we see that curl(F) = div(G). Green's theorem now becomes

$$\int\int_R \text{div}(G)\ dxdy = \int_\gamma G\cdot dn$$

where $dn(x,y)$ is a normal vector at (x,y) orthogonal to the velocity vector $r'(x,y)$ at (x,y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. **In two dimensions it is just Green's theorem disguised.** There are only 2 basic integral theorems in the plane: Green's theorem and the FTLI. This version of Greens theorem however tells us what **divergence** is in two dimensions.

OVERVIEW OF DERIVATIVES: in dimension d, there are d fundamental derivatives:

$1\mapsto 1$	f'	derivative

$1\mapsto 2$	∇f	gradient
$2\mapsto 1$	$\nabla\times F$	curl

$1\mapsto 3$	∇f	gradient
$3\mapsto 3$	$\nabla\times F$	curl
$3\mapsto 1$	$\nabla\cdot F$	divergence