## CHAIN RULE — Math21a, O. Knill

HOMEWORK Section 13.5: 2,6, 12, 44, 46

1D CHAIN RULE. If $f$ and $g$ are functions of one variable $t$, then $d/dt f(g(t)) = f'(g(t))g'(t)$. For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$.

FINDING DERIVATIVES. Also the 1D chain rule was useful. For example, to find the derivative of $\log(x)$ we can write $1 = d/dx \exp(log(x) = d/dx \exp(\log(x)) = \log'(x)x$ so that $\log'(x) = 1/x$. An other example: to find $\arccos'(x)$, we write $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x))\arccos'(x) = -\sqrt{1-\sin^2(\arccos(x))}\arccos'(x) = \sqrt{1-x^2}\arccos'(x)$ so that $\arccos'(x) = -1/\sqrt{1-x^2}$.

GRADIENT. Define $\nabla f(x,y) = (f_x(x,y), f_y(x,y))$. It is called the **gradient** of $f$. It is the natural derivative of a function of several variables and a vector.

THE CHAIN RULE. If $\vec{r}(t)$ is curve in space and $f$ is a function of three variables, we get a function of one variables $t \mapsto f(\vec{r}(t))$. The **chain rule** is

$$\boxed{d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)}$$

WRITING IT OUT. The chain rule is, when written out

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) .$$

EXAMPLE. Let $z = \sin(x+2y)$, where $x$ and $y$ are functions of $t$: $x = e^t, y = \cos(t)$. What is $\frac{dz}{dt}$?

Here, $z = f(x,y) = \sin(x+2y)$, $z_x = \cos(x+2y)$, and $z_y = 2\cos(x+2y)$ and $\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = -\sin(t)$ and $\frac{dz}{dt} = \cos(x+2y)e^t - 2\cos(x+2y)\sin(t)$.

EXAMPLE. If $f$ is the temperature distribution in a room and $\vec{r}(t)$ is the path of the spider **Shelob**, then $f(\vec{r}(t))$ is the temperature, Shelob experiences at time $t$. The rate of change depends on the velocity $\vec{r}'(t)$ of the spider as well as the temperature gradient $\nabla f$ and the angle between gradient and velocity. For example, if the spider moves perpendicular to the gradient, its velocity is tangent to a level curve and the rate of change is zero.

EXAMPLE. Olivers spider "Nabla" moves along a circle $\vec{r}(t) = (\cos(t), \sin(t))$ on a table with temperature distribution $T(x,y) = x^2 - y^3$. Find the rate of change of the temperature, "Nabla" experiences.

SOLUTION. $\nabla T(x,y) = (2x, -3y^2)$, $\vec{r}'(t) = (-\sin(t), \cos(t))$ $d/dt T(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = (2\cos(t), -3\sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.

APPLICATION ENERGY CONSERVATION. If $H(x,y)$ is the energy of a particle with position $x$ and velocity $y$, the system moves satisfies the equations $\boxed{x'(t) = H_y, y'(t) = -H_x}$. For example, if $H(x,y) = y^2/2 + V(x)$ is a sum of kinetic and potential energy, then $x'(t) = y, y'(t) = V'(x)$ is equivalent to $x''(t) = -V'(x)$. **[** THEOREM: The energy $H$ is conserved **]**. Proof. The chain rule shows that $d/dt H(x(t), y(t)) = H_x(x,y)x'(t) + H_y(x,y)y'(t) = H_x(x,y)H_y(x,y) - H_y(x,y)H_x(x,y) = 0$.

APPLICATION: IMPLICIT DIFFERENTIATION.
From $f(x,y) = 0$ one can express $y$ as a function of $x$. From $d/df(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$, we obtain

$$\boxed{y' = -f_x/f_y}.$$

Even so, we do not know $y(x)$, we can compute its derivative!

EXAMPLE. $f(x,y) = x^4 + x\sin(xy) = 0$ defines $y = g(x)$. If $f(x, g(x)) = 0$, then $g_x(x) = -f_x/f_y = -(4x^3 + \sin(xy) + xy\cos(xy))/(x^2\cos(xy))$.

IMPLICIT DIFFERENTIATION IN THREE VARIABLES. The equation $f(x,y,z) = c$ defines a surface. Near a point where $f_z$ is not zero, the surface can be described as a graph $z = z(x,y)$. We can compute the derivative $z_x$ without actually knowing the function $z(x,y)$! To do so, we consider $y$ a fixed parameter and compute using the chain rule

$$f_x(x,y,z(x,y))1 + f_z(x,y)z_x(x,y) = 0$$

so that $z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$.

EXAMPLE. Let $f(x,y,z) = x^2 + y^2/4 + z^2/9 = 6$ be an ellipsoid. Compute $z_x(x,y)$ at the point $(x,y,z) = (2,1,1)$.
Solution: $z_x(x,y) = -f_x(2,1,1)/f_z(2,1,1) = -4/(2/9) = -18$.

APPLICATION: DIFFERENTIATION RULES. One ring of the chain rules them all!
$f(x,y) = x+y, x = u(t), y = v(t), d/dt(x+y) = f_x u' + f_y v' = u' + v'$.
$f(x,y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x,y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.

DIETERICI EQUATION. In thermodynamics the temperature $T$, the pressure $p$ and the volume $V$ of a gas are related. One refinement of the ideal gas law $pV = RT$ is the **Dieterici equation** $f(p,V,T) = p(V-b)e^{a/RVT} - RT = 0$. The constant $b$ depends on the volume of the molecules and $a$ depends on the interaction of the molecules. (A different variation of the ideal gas law is the van der Waals law). Problem: compute $V_T$.

If $V = V(T,p)$, the chain rule says $f_V V_T + f_T = 0$, so that $\boxed{V_T = -f_T/f_V} = -(-ap(V-b)e^{a/RVT}/(RVT^2) - R)/(pVe^{a/RVT} - p(V-b)e^{a/RVT}/(RV^2T))$. (This could be simplified to $(R + a/TV)/(RT/(V-b) - a/V^2)$).

PROOF OF THE CHAIN RULE.

Near any point, we can approximate $f$ by a linear function $L$. It is enough to check the chain rule for linear functions $f(x,y) = ax + by - c$ and if $\vec{r}(t) = (x_0 + tu, y_0 + tv)$ is a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv$ and this is the dot product of $\nabla f = (a,b)$ with $\vec{r}\,'(t) = (u,v)$.

WHERE IS THE CHAIN RULE NEEDED?

While the chain rule is useful in calculations using the composition of functions, the iteration of maps or in doing change of variables, it is also useful for **understanding** some theoretical aspects. Examples:

- In the proof of the fact that **gradients are orthogonal to level surfaces**. (see the Wednesday lecture).
- It appears in **change of variable** formulas.
- It will be used in the **fundamental theorem for line integrals** coming up later in the course.
- The chain rule illustrates also the **Lagrange multiplier** method which we will see later.
- In **fluid dynamics**, PDE's often involve terms $u_t + u\nabla u$ which give the change of the velocity in the frame of a fluid particle.
- In **chaos theory**, where one wants to understand what happens after iterating a map $f$.

APPLICATION: If $f(x,y,z) = 0$, then $x = x(y,z), y = y(x,z)$ and $z = z(x,y)$. From $y_x = -f_x/f_y, x_z = -f_z/f_x, z_y = -f_y/f_z$ we get the relation $\boxed{y_x x_z z_y = -1}$. This formula appears in thermodynamics.

EXAMPLE. GRADIENT IN POLAR COORDINATES. In polar coordinates, the gradient is defined as $\nabla f = (f_r, f_\theta/r)$. Using the chain rule, we can relate this to the usual gradient: $d/dr f(x(r,\theta), y(r,\theta)) = f_x(x,y)\cos(\theta) + f_y(x,y)\sin(\theta)$ and $d/(rd\theta) f(x(r,\theta), y(r,\theta)) = -f_x(x,y)\sin(\theta) + f_y(x,y)\cos(\theta)$ means that the length of $\nabla f$ is the same in both coordinate systems.