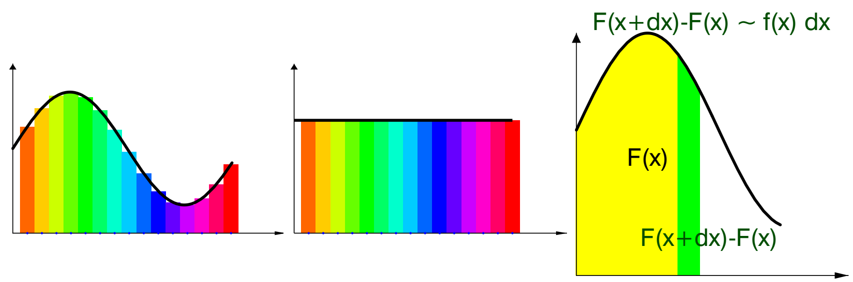
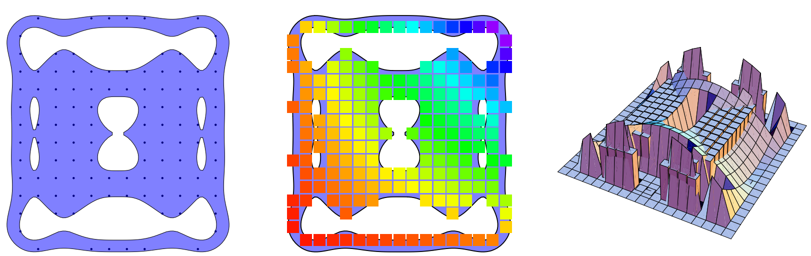
2D INTEGRALS — Math21a, O. Knill

HOMEWORK: 14.1: 18, 44 14.2: 24,26 14.3: 26

1D INTEGRATION IN 100 WORDS. If $f(x)$ is a continuous function, then $\int_a^b f(x)\,dx$ can be defined as a limit of the **Riemann sum** $f_n(x) = \sum_{x_k \in [a,b]} f(x_k)\Delta x$ for $n \to \infty$ with $x_k = k/n$ and $\Delta x = 1/n$. This integral divided by $|b-a|$ is the **average** of f on $[a,b]$. The integral $\int_a^b f(x)\,dx$ can be interpreted as an **signed area** under the graph of f, which can be negative too. If $f(x) = 1$, the integral is the **length** of the interval. The function $F(x) = \int_a^x f(y)\,dy$ is called an **anti-derivative** of f. The **fundamental theorem of calculus** states $F'(x) = f(x)$. This allows to compute integrals by inverting differentiation. Differentiation rules like the Leibnitz rule become integration rules like integration by part, the chain rule becomes partial integration. Note that unlike the derivative, anti-derivatives can not always be expressed in terms of known functions. An example is: $F(x) = \int_0^x e^{-t^2}\,dt$. Often, the anti-derivative can be found: Example: $f(x) = \cos^2(x) = (\cos(2x)+1)/2, F(x) = x/2 - \sin(2x)/4$.



2D INTEGRATION. If $f(x,y)$ is a continuous function of two variables on a region R, the integral $\int_R f(x,y)\,dxdy$ can be defined as the limit $\sum_{i,j,x_{i,j} \in R} f(x_i,y_j)\Delta x \Delta y$ with $x_{i,j} = (i/n, j/n)$ when n goes to infinity. If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral divided by the area of R is the **average** value of f on R. For many regions, the integral can be calculated as a **double integral** $\int_a^b [\int_{c(x)}^{d(x)} f(x,y)\,dy]dx$. In general, the region must be split into pieces, then integrated seperately.



One can interpret $\int\int_R f(x,y)\,dydx$ as the volume of solid below the graph of f and above R in the $x-y$ plane.

As in 1D integration, the volume of the solid below the x-y plane is counted negatively.

EXAMPLE. Calculate $\int\int_R f(x,y)\,dxdy$, where $f(x,y) = 4x^2y^3$ and where R is the rectangle $[0,1] \times [0,2]$.

$$\int_0^1 [\int_0^2 4x^2y^3\,dy]\,dx = \int_0^1 [x^2y^4|_0^2]\,dx = \int_0^1 x^2(16-0)\,dx = 16x^3/3|_0^1 = \frac{16}{3} .$$

FUBINI'S THEOREM.

$$\int_a^b \int_c^d f(x,y)\,dxdy = \int_c^d \int_a^b f(y,x)\,dydx.$$

PROOF. Approximate both sides with a Riemann sum, where $\Delta x = \Delta y = 1/n$. We have the identity

$$\sum_{x_i \in [a,b]} \sum_{y_j \in [c,d]} f(x_i,y_j)\,\Delta y \Delta y = \sum_{y_j \in [c,d]} \sum_{x_i \in [a,b]} f(x_i,y_j)\,\Delta x \Delta x .$$

Now take the limit $n \to \infty$.

EXAMPLE FROM QUANTUM MECHANICS. In quantum mechanics, the motion of a particle (like an electron) in the plane is determined by a function $u(x,y)$, the wave function. Unlike in classical mechanics, the position of a particle is given in a probabilistic way only. If R is a region and u is normalized so that $\int |u|^2\,dxdy = 1$, then $\int_R |u(x,y)|^2\,dxdy$ is the **probability**, that the particle is in R.

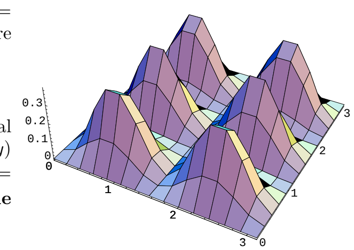
EXAMPLE. Unlike a classical particle, a quantum particle in a box $[0,\pi] \times [0,\pi]$ can have a discrete set of energies only. This is the reason for the name "quantum". If u satisfies the PDE

$$-(u_{xx} + u_{yy}) = \lambda u ,$$

then a particle of mass m has the energy $E = \lambda \hbar^2/2m$. A function $u(x,y) = \sin(kx)\sin(ny)$ represents a particle of energy $(k^2+n^2)\hbar^2/(2m)$. Let us assume $k = 2$ and $n = 3$ from now on. Our aim is to find the probability that the particle with energy $13\hbar^2/(2m)$ is in the middle 9'th $R = [\pi/3, 2\pi/3] \times [\pi/3, 2\pi/3]$ of the box.

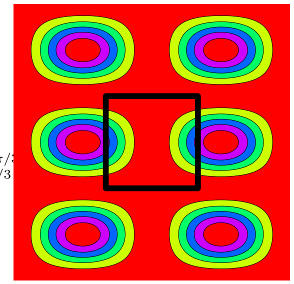
SOLUTION: We first have to normalize $u^2(x,y) = \sin^2(2x)\sin^2(3y)$, so that the average over the whole square is 1:

$$A = \int_0^\pi \int_0^\pi \sin^2(2x)\sin^2(3y)\,dxdy .$$



To calculate this integral, we first determine the inner integral $\int_0^\pi \sin^2(2x)\sin^2(3y)\,dx = \sin^2(3y)\int_0^\pi \sin^2(2x)\,dx = \frac{\pi}{2}\sin^2(3y)$ (the factor $\sin^2(3y)$ is treated as a constant). Now, $A = \int_0^\pi (\pi/2)\sin^2(3y)\,dy = \frac{\pi^2}{4}$, so that the **probability amplitude function** is $f(x,y) = \frac{4}{\pi^2}\sin^2(2x)\sin^2(3y)$.

The probability that the particle is in R is slightly smaller than 1/9:



$$\begin{aligned} \frac{1}{A}\int_R f(x,y)\,dxdy &= \frac{4}{\pi^2}\int_{\pi/3}^{2\pi/3}\int_{\pi/3}^{2\pi/3} \sin^2(2x)\sin^2(3y)\,dxdy \\ &= \frac{4}{\pi^2}(4x - \sin(4x))/8|_{\pi/3}^{2\pi/3}(6x - \sin(6x))/12|_{\pi/3}^{2\pi/3} \\ &= 1/9 - 1/(4\sqrt{3}\pi) \end{aligned}$$

The probability is slightly smaller than 1/9.

WHERE DO DOUBLE INTEGRALS OCCUR?

- compute areas.
- compute averages. Examples: average rain fall or average population in some area.
- probabilities. Expectation or variance of random variables.
- quantum mechanics: probability of particle being in a region.
- find the moment of inertia $\int\int_R (x^2+y^2)\rho(x,y)dxdy$
- find the center of mass $(\int\int_R x\rho(x,y)\,dxdy/M, \int\int_R y\rho(x,y)\,dxdy/M)$, with $M = \int\int_R dxdy$.
- compute some 1D integrals like $\int_\infty^\infty e^{-x^2}\,dx$.