

GLOSSARY CHECKLIST

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Geometry of Space

coordinates and vectors in the plane and in space

sum $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3), v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

dot product $v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3 = |v||w| \cos(\alpha)$

cross product, $v \cdot (v \times w) = 0, w \cdot (v \times w) = 0, |v \times w| = |v||w| \sin(\alpha)$

triple scalar product $u \cdot (v \times w)$ volume of parallelepiped

parallel vectors $v \times w = 0$, orthogonal vectors $v \cdot w = 0$

scalar projection $\text{comp}_w(v) = v \cdot w / |w|$

vector projection $\text{proj}_w(v) = (v \cdot w / |w|^2) w$

completion of square: example $x^2 - 4x + y^2 = 1$ is equivalent to $(x - 2)^2 + y^2 = -3$

distance $d(P, Q) = |PQ| = \sqrt{(P_1 - Q_1)^2 + (P_2 - Q_2)^2 + (P_3 - Q_3)^2}$

Lines, Planes, Functions

symmetric equation of line $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{(z-z_0)}{c}$
 plane $ax + by + cz = d$
 parametric equation for line $\vec{x} = \vec{x}_0 + t\vec{v}$
 parametric equation for plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$
 switch from parametric to implicit descriptions for lines and planes
 domain and range of functions $f(x, y)$
 graph $G = \{(x, y, f(x, y))\}$
 intercepts: intersections of G with coordinate axis
 traces: intersections with coordinate planes
 generalized traces: intersections with $\{x = c\}$, $\{y = c\}$ or $\{z = c\}$
 quadrics: ellipsoid, paraboloid, hyperboloids, cylinder, cone, hyperboloid paraboloid
 plane $ax + by + cz = d$ has normal $\vec{n} = (a, b, c)$
 line $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{(z-z_0)}{c}$ contains $\vec{v} = (a, b, c)$
 sets $g(x, y, z) = c$ describe surfaces, example graphs $g(x, y, z) = z - f(x, y)$
 linear equation like $2x + 3y + 5z = 7$ defines plane
 quadratic equation like $x^2 - 2y^2 + 3z^2 = 4$ defines quadric surface
 distance point-plane: $d(P, \Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$
 distance point-line: $d(P, L) = |(\vec{PQ}) \times \vec{w}|/|\vec{w}|$
 distance line-line: $d(L, M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$
 finding plane through three points A, B, C : find normal vector $(a, b, c) = \vec{AB} \times \vec{CB}$

Curves

plane and space curves $\vec{r}(t)$
velocity $\vec{r}'(t)$, acceleration $\vec{r}''(t)$
unit tangent vector $\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$
 $\vec{r}'(t)$ is tangent to the curve
 $\vec{v} = \vec{r}'$ then $\vec{r} = \int_0^t \vec{v} dt + \vec{c}$
 $\vec{r}(t) = (f(t)\cos(t), r(t)\sin(t))$ polar curve to polar graph $r = f(\theta)$.
 $\kappa(t) = |\vec{T}'(t)|/|\vec{r}'(t)|$ curvature

Surfaces

polar coordinates $(x, y) = (r \cos(\theta), r \sin(\theta))$
cylindrical coordinates $(x, y, z) = (r \cos(\theta), r \sin(\theta), z)$
spherical coordinates $(x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi))$
 $g(r, \theta) = 0$ polar curve, especially $r = f(\theta)$, polar graphs
 $g(r, \theta, z) = 0$ cylindrical surface, i.e. $r = f(z, \theta)$ or $r = f(z)$ surface of revolution
 $g(\rho, \theta, \phi) = 0$ spherical surface especially $\rho = f(\theta, \phi)$
 $f(x, y) = c$ level curves of $f(x, y)$, normal vectors are $\nabla f(x, y)$
 $g(x, y, z) = c$ level surfaces of $g(x, y, z)$, normal vectors are $\nabla f(x, y, z)$
circle: $x^2 + y^2 = r^2$, $\vec{r}(t) = (r \cos t, r \sin t)$.
ellipse: $x^2/a^2 + y^2/b^2 = 1$, $\vec{r}(t) = (a \cos t, b \sin t)$
sphere: $x^2 + y^2 + z^2 = r^2$, $\vec{r}(u, v) = (r \cos u \sin v, r \sin u \sin v, r \cos v)$
ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $\vec{r}(u, v) = (a \cos u \sin v, b \sin u \sin v, c \cos v)$
line: $ax + by = d$, $\vec{r}(t) = (t, d/b - ta/b)$
plane: $ax + by + cz = d$, $\vec{r}(u, v) = \vec{r}_0 + u\vec{v} + v\vec{w}$, $(a, b, c) = \vec{v} \times \vec{w}$
surface of revolution: $r(\theta, z) = f(z)$, $\vec{r}(u, v) = (f(v) \cos(u), f(v) \sin(u), v)$
graph: $g(x, y, z) = z - f(x, y) = 0$, $\vec{r}(u, v) = (u, v, f(u, v))$

Partial Derivatives

$f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$ partial derivative
 partial differential equation PDE: $F(f, f_x, f_t, f_{xx}, f_{tt}) = 0$
 $f_t = f_{xx}$ heat equation
 $f_{tt} - f_{xx} = 0$ wave equation
 $f_x - f_t = 0$ transport equation
 $f_{xx} + f_{yy} = 0$ Laplace equation
 $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ linear approximation
 tangent line: $L(x, y) = L(x_0, y_0)$, $ax + by = d$ with $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $d = ax_0 + by_0$
 tangent plane: $L(x, y, z) = L(x_0, y_0, z_0)$
 estimate $f(x, y, z)$ by $L(x, y, z)$ near (x_0, y_0, z_0)
 $|f(x, y) - L(x, y)|$ in box R around (x_0, y_0) is $\leq M(|x - x_0| + |y - y_0|)^2/2$, where M is the maximal value of $|f_{xx}(x, y)|, |f_{xy}(x, y)|, |f_{yy}(x, y)|$ in R .
 $f(x, y)$ called differentiable if f_x, f_y are continuous
 $f_{xy} = f_{yx}$ Clairot's theorem
 $\vec{r}_u(u, v), \vec{r}_v$ tangent to surface $\vec{r}(u, v)$

Gradient

$\nabla f(x, y) = (f_x, f_y), \nabla f(x, y, z) = (f_x, f_y, f_z), \text{ gradient}$
 $D_v f = \nabla f \cdot v \text{ directional derivative}$
 $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \text{ chain rule}$
 $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x, y, z) = c$ which contains (x_0, y_0, z_0) .
 $\frac{d}{dt} f(\vec{x} + t\vec{v}) = D_v f$ by chain rule
 $\frac{x-x_0}{f_x(x_0, y_0, z_0)} = \frac{y-y_0}{f_y(x_0, y_0, z_0)} = \frac{z-z_0}{f_z(x_0, y_0, z_0)}$ normal line to surface $f(x, y, z) = c$ at (x_0, y_0, z_0)
 $(x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) + (z - z_0)f_z(x_0, y_0, z_0) = 0$ tangent plane at (x_0, y_0, z_0)
 directional derivative is maximal in the $\vec{v} = \nabla f$ direction
 $f(x, y)$ increases, if we walk on the xy -plane in the ∇f direction
 partial derivatives are special directional derivatives
 if $D_v f(\vec{x}) = 0$ for all $\vec{v}$, then $\nabla f(\vec{x}) = \vec{0}$

Extrema

$\nabla f(x, y) = (0, 0)$, critical point or stationary point
 $D = f_{xx}f_{yy} - f_{xy}^2$ discriminant or Hessian determinant
 $f(x_0, y_0) \geq f(x, y)$ in a neighborhood of (x_0, y_0) local maximum
 $f(x_0, y_0) \leq f(x, y)$ in a neighborhood of (x_0, y_0) local minimum
 $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = c, \lambda$ Lagrange equations
 $\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = c, \lambda$ Lagrange equations
second derivative test: $\nabla f = (0, 0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0, 0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0, 0), D < 0$ **saddle**

Double Integrals

$\int \int_R f(x, y) dA$ double integral
 $\int_a^b \int_c^d f(x, y) dy dx$ integral over rectangle
 $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$ type I region
 $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$ type II region
 $\int \int_R f(r, \theta) r dr d\theta$ polar coordinates
 $\int \int_R |\vec{r}_u \times \vec{r}_v| du dv$ surface area
 $\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$ Fubini
 $\int \int_R 1 dx dy$ area of region R
 $\int \int_R f(x, y) dx dy$ volume of solid bounded by graph(f) xy-plane

Triple Integrals

$\int \int \int_R f(x, y, z) dV$ triple integral
 $\int_a^b \int_c^d \int_u^v f(x, y, z) dy dx dz$ integral over rectangular box
 $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y) dz dy dx$ type I region
 $f(r, \theta, z) r dz dr d\theta$ cylindrical coordinates
 $\int \int \int_R f(\rho, \theta, z) \rho^2 \sin(\phi) dz dr d\theta$ spherical coordinates
 $\int_a^b \int_c^d \int_u^v f(x, y, z) dz dy dx = \int_u^v \int_c^d \int_a^b f(x, y, z) dx dy dz$ Fubini
 $V = \int \int \int_R 1 dV$ volume of solid R
 $M = \int \int \int_R \rho(x, y, z) dV$ mass of solid R with density ρ
 $(\int \int \int_R x dV/V, \int \int \int_R y dV/V, \int \int \int_R z dV/V)$ center of mass

Line Integrals

$F(x, y) = (P(x, y), Q(x, y))$ vector field in the plane
 $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$ vector field in space
 $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt$ line integral
Use also notation $\int_C F \cdot T ds$ and $\int P dx + Q dy + R dz$.
 $F(x, y) = \nabla f(x, y)$ gradient field = potential field = conservative

Fundamental theorem of line integrals

FTL: $F(x, y) = \nabla f(x, y), \int_a^b F(r(t)) \cdot r'(t) dt = f(r(b)) - f(r(a))$
Closed loop property $\int_C F dr = 0$, for all closed curves C
Always equivalent are: closed loop property, conservativeness and gradient field
Mixed derivative test $\text{curl}(F) \neq 0$ assures F is not a gradient field.
In simply connected domains, $\text{curl}(F) = 0$ implies conservativeness.

Green's Theorem

$F(x, y) = (P, Q)$, curl in two dimensions: $\text{curl}(F) = Q_x - P_y = \nabla \times F$.
Green's theorem: C boundary of R , then $\int_C F \cdot dr = \int \int_R \text{curl}(F) dx dy$
Area computation: Take F with $\text{curl}(F) = Q_x - P_y = 1$ like $F = (-y, 0)$ or $F = (0, x)$ or $F = (-y, x)/2$.
Greens theorem is useful to compute difficult line integrals or difficult 2D integrals.

Flux integrals

$F(x, y, z)$ vector field, $S = r(R)$ parametrized surface
 $r_u \times r_v$ normal vector, $\vec{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$ unit normal vector
 $r_u \times r_v du dv = d\vec{S} = \vec{n} dS$ normal surface element
 $\int \int_S F d\vec{S} = \int \int_S F(r(u, v)) \cdot (r_u \times r_v) du dv$ flux integral
Use also notation $\int \int_S F \cdot n d\sigma$

Stokes Theorem

$F(x, y, z) = (P, Q, R)$, $\text{curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \nabla \times F$
Stokes's theorem: C boundary of surface S , then $\int_C F \cdot dr = \int \int_S \text{curl}(F) \cdot d\vec{S}$
Stokes theorem is useful to compute difficult flux integrals of $\text{curl}(F)$ or difficult line integrals.

Div Grad Curl

$\nabla = (\partial_x, \partial_y, \partial_z)$, $\text{grad}(F) = \nabla f, \text{curl}(F) = \nabla \times F, \text{div}(F) = \nabla \cdot F$
 $\text{div}(\text{curl}(F)) = 0$
 $\text{curl}(\text{grad}(F)) = \vec{0}$
 $\text{div}(\text{grad}(f)) = \Delta f$.

Divergence Theorem

$\text{div}(P, Q, R) = P_x + Q_y + R_z = \nabla \cdot F$
Divergence theorem: solid E , boundary S then $\int \int_S F \cdot d\vec{S} = \int \int \int_E \text{div}(F) dV$
The divergence theorem is useful to compute difficult flux integrals or difficult 3D integrals.

Some topology

Simply connected region D : can deform any closed curve within D to a point on curve.
Interior of a region D : points in D for which small neighborhood is still in D .
Boundary of a curve: the end points of the curve if they exist.
Boundary of a surface S are curves which bound the surface, points in the surface which correspond to parameters (u, v) which are not in the interior of the parametrization domain.
Boundary of a solid D : the surfaces which bound the solid, points in the solid which are not in the interior of D .
Closed surface: a surface without boundary like for example the sphere.
Closed curve: a curve with no boundary like for example a knot.

Some surface parametrizations

Sphere of radius ρ : $r(u, v) = (\rho \cos(u) \sin(v), \rho \sin(u) \sin(v), \rho \cos(v))$
Graph of function $f(x, y)$: $f(u, v) = (u, v, f(u, v))$
Graph of function $f(\phi, r)$ in polar: $f(u, v) = (v \cos(u), v \sin(u), f(u, v))$
Plane containing P and vectors $\vec{u}, \vec{v}$: $r(u, v) = P + u\vec{u} + v\vec{v}$
Surface of revolution: distance $g(z)$ of z -axis: $r(u, v) = (g(v) \cos(u), g(v) \sin(u), v)$
Ex: Cylinder: $r(u, v) = (\cos(u), \sin(u), v)$.
Ex: Cone: $r(u, v) = (v \cos(u), v \sin(u), v)$.
Ex: Paraboloid: $r(u, v) = (\sqrt{r} \cos(u), \sqrt{r} \sin(u), v)$.