

## 1. TRIGONOMETRIC IDENTITIES

The Pythagorean Theorem,  $\sin^2 x + \cos^2 x = 1$ , has other forms, like

$$(1) \quad \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$$

and

$$(2) \quad \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 1 + \cot^2 x = \csc^2 x.$$

Angle Addition:

$$(3) \quad \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$(4) \quad \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

The angle addition formulae give us the double angle formulae:

$$(5) \quad \sin(2\theta) = 2 \sin \theta \cos \theta$$

$$(6) \quad \cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

Because  $\cos(2\theta) = \cos^2 \theta - (1 - \cos^2 \theta) = 2 \cos^2 \theta - 1$ , we have the half-angle formula

$$(7) \quad \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

Similarly, because  $\cos(2\theta) = 1 - \sin^2 \theta - \sin^2 \theta$ , we find the other half-angle formula

$$(8) \quad \sin^2 \theta = \frac{1 - \cos(2\theta)}{2}.$$

## 2. DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

We know the derivatives

$$(9) \quad \frac{d}{dx} \sin x = \cos x \quad \text{and} \quad \frac{d}{dx} \cos x = -\sin x,$$

which allow us to calculate

$$(10) \quad \frac{d}{dx} \tan x = \frac{\cos x}{\cos x} + \sin x \frac{\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

$$(11) \quad \frac{d}{dx} \csc x = \frac{-\cos x}{\sin^2 x} = -\csc x \cot x$$

$$(12) \quad \frac{d}{dx} \sec x = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

$$(13) \quad \frac{d}{dx} \cot x = \frac{-\sin x}{\sin x} + \cos x \frac{-\cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x$$

## 3. USEFUL INTEGRALS

To compute  $\int \cos^2 x dx$ , we use the half-angle formula for cosines:

$$(14) \quad \int \cos^2 x dx = \int \frac{1 + \cos(2x)}{2} dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$$

Similarly, to compute  $\int \sin^2 x dx$ , we use the half-angle formula for sines:

$$(15) \quad \int \sin^2 x dx = \int \frac{1 - \cos(2x)}{2} dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

To compute  $\int \frac{dx}{1-x^2}$ , we use the method of partial fractions:

$$(16) \quad \int \frac{dx}{1-x^2} = \frac{1}{2} \int \left( \frac{1}{1+x} + \frac{1}{1-x} \right) dx = \frac{1}{2} [\ln(1+x) - \ln(1-x)] + C$$

## 4. INTEGRALS USING TRIGONOMETRIC SUBSTITUTIONS

To compute  $\int \frac{dx}{1+x^2}$ , we substitute

$$(17) \quad x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

to find

$$(18) \quad \int \frac{dx}{1+x^2} = \int \frac{\sec^2 \theta}{\sec^2 \theta} d\theta = \theta + C = \arctan x + C$$

To compute  $\int \frac{dx}{\sqrt{1-x^2}}$ , we substitute

$$(19) \quad x = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

to find

$$(20) \quad \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos \theta}{\sqrt{1-\sin^2 \theta}} d\theta = \theta + C = \arcsin x + C$$

## 5. INTEGRATION BY PARTS

The Product Rule states that  $d(uv) = u dv + v du$ . Integrating, we find

$$(21) \quad \int d(uv) = \int u dv + \int v du \Rightarrow uv = \int u dv + \int v du \Rightarrow \int u dv = uv - \int v du.$$

For example, to integrate  $x e^x$ , we choose  $u = x$  and  $v = e^x$ , so that  $du = dx$  and  $dv = e^x$ , and calculate

$$(22) \quad \int x e^x dx = x e^x + C' - \int e^x dx = x e^x - e^x + C$$