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- Please mark the box to the left which lists your section.
- Do not detach pages from this exam packet or unstaple the packet.
- Show your work. Answers without reasoning can not be given credit.
- Please write neatly. Answers which the grader can not read will not receive credit except for the True/False and multiple choice problems.
- No notes, books, calculators, computers, or other electronic aids can be used.
- All unspecified functions mentioned in this exam are assumed to be smooth: you can differentiate as many times as you want with respect to any variables.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) **T** F — If $\nabla f(x, y) \neq \langle 0, 0 \rangle$ at a given point (x_0, y_0), there exists a unit vector $\vec{u}$ for which $D_{\vec{u}} f(x_0, y_0)$ is zero.

Solution:
Take a vector which is tangent to the level curve. Because this vector is perpendicular to the gradient, the directional derivative $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ is zero.

2) **T** F — If $f_{xx}(0, 0) = 0$, $D \neq 0$, and $\nabla f(0, 0) = \langle 0, 0 \rangle$, then $(0, 0)$ is a saddle point.

Solution:
Because $f_{xx} = 0$, we have $D = f_{xx} f_{yy} - f_{xy}^2 = -f_{xy}^2$ which can not be positive. Because $D \neq 0$, we must have $D < 0$. By the second derivative test, the critical point is a saddle point.

3) **T** F — The surface described in spherical coordinates by the equation $\rho \cos(\phi) = \rho^2 \sin^2(\phi)$ is an elliptic paraboloid.

Solution:
The equation means in cylindrical coordinates $z = r^2$. In Cartesian coordinates, this is $z = x^2 + y^2$.

4) T **F** — The function $f(x, t) = x + t$ satisfies the heat equation $f_t = f_{xx}$.

Solution:
$f_t = 1$ but $f_{xx} = 0$.

5) **T** F — $f(x, y) = 3x^2 y - y^3$ is a solution of the Laplace equation $f_{xx} + f_{yy} = 0$.

Solution:
$f_{xx} + f_{yy} = 0$.

6) T F

A smooth function defined on the closed unit disc $x^2+y^2 \leq 1$ has an absolute maximum in this disc (including the boundary).

Solution:
The maximum can be either in the interior or at the boundary.

7) T F

A surface defined in cylindrical coordinates by the equation $g(r, \theta, z) = 0$ is always a surface of revolution.

Solution:
It is a surface of revolution if there is no θ dependence.

8) T F

The function $f(x, y) = x^2 - y^2$ has a neither a local maximum nor a local minimum at $(0, 0)$.

Solution:
It is a saddle point.

9) T F

The functions $f(x, y)$ and $g(x, y) = (f(x, y))^4$ always have the same critical points.

Solution:
The gradient of g is $4f^3(x, y)\nabla f$. So, the second function has critical points, where f vanishes.

10) T F

For $f(x, y, z) = x^2 + y^2 + 2z^2$, the vector $\nabla f(1, 1, 1)$ is perpendicular to the surface $f(x, y, z) = 4$ at the point $(1, 1, 1)$.

Solution:
This is a basic property of gradients.

11) T F

If $f(x, y) = c$ and $f_x \neq 0$, then $\frac{dx}{dy} = f_y(x, y)/f_x(x, y)$.

Solution:
This is a formula for implicit differentiation. It is almost right, but the sign is wrong.

12) T F $f(x, y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition.

Solution:
The domain of definition is the disc $x^2 + y^2 \leq 16$. The maximum 4 is in the center the absolute minimum 0 at the boundary.

13) T F If (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f.

Solution:
The discriminant $D = f_{xx} f_{yy} - f_{xy}^2$ can be positive. An example is $f(x, y) = 100x^2 + 100y^2 - xy$.

14) T F The vector $\vec{r}_v(u, v)$ of a parameterized surface $(u, v) \mapsto \vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ is always perpendicular to the surface.

Solution:
The vector is always tangent to the surface.

15) T F The directional derivative $D_{\vec{v}} f$ is a vector perpendicular to $\vec{v}$.

Solution:
The directional derivative is a number, not a vector.

16) T F Suppose f has a maximum value at a point P relative to the constraint $g = 0$. If the Lagrange multiplier $\lambda = 0$, then P is also a critical point for f without the constraint.

Solution:
The Lagrange equations tell that $\nabla f(x, y) = (0, 0)$.

17) T F At a saddle point, all directional derivatives are zero.

Solution:
Because $\nabla f(x,y) = (0,0)$ at a saddle point, all directional derivatives $D_{\vec{v}}f = \nabla f \cdot \vec{v}$ are zero.

18) | T | F |

The minimum of $f(x,y)$ under the constraint $g(x,y) = 0$ is always the same as the maximum of $g(x,y)$ under the constraint $f(x,y) = 0$.

Solution:
This can not be true, because the first problem is the same if we replace $g(x,y)$ with $2g(x,y)$, but this will change the value of the maximum of g on the right hand side.

19) | T | F |

The function $f(t,x,y) = y\sin(x-t)$ satisfies the partial differential equation $f_{tt} = f_{xx} + f_{yy}$.

Solution:
Indeed this is a solution. Note that $f_{yy} = 0$ and $f_{tt} = f_{xx}$. The partial differential equation is called two dimensional wave equation.

20) | T | F |

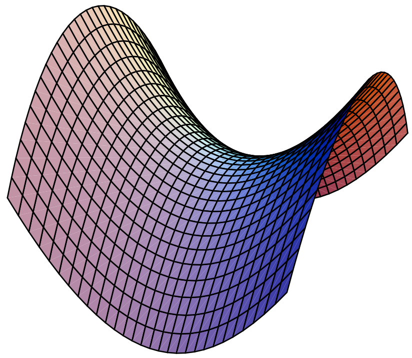
At a local maximum (x_0, y_0) of $f(x,y)$, one has $f_{yy}(x_0,y_0) \leq 0$.

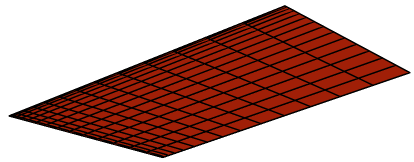
Solution:
Indeed, at a local maximum, $f_{yy} \leq 0$.

Problem 2) (10 points)

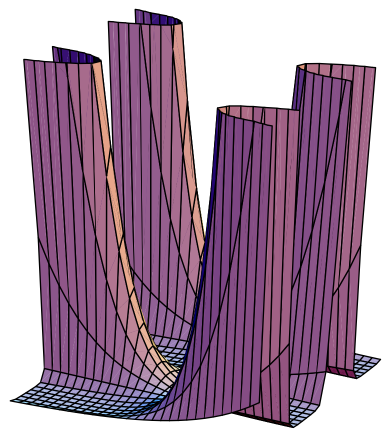
Match the parametric surfaces with their parameterization. No justifications are needed in this problem.

I II

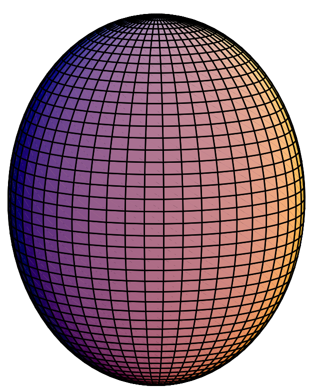




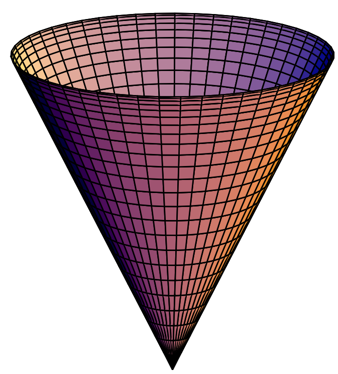
III



IV



V



Enter I,II,III,IV here	Parameterization
	$(u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), 5\cos(v))$
	$(u,v) \mapsto (u^2, v^2, u^2 - v^2)$
	$(u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v), 5\sin(v))$
	$(u,v) \mapsto (u, v, u^2 - v^2)$
	$(u,v) \mapsto (u, v, e^{u\sin(v)})$

Solution:

Enter I,II,III,IV here	Parameterization
IV	$(u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v)/2, 5\cos(v))$
II	$(u,v) \mapsto (u^2, v^2, u^2 - v^2)$
V	$(u,v) \mapsto (\cos(u)\sin(v), \sin(u)\sin(v)/2, 5\sin(v))$
I	$(u,v) \mapsto (u, v, u^2 - v^2)$
III	$(u,v) \mapsto (u, v, e^{u\sin(v)})$

Problem 3) (10 points)

Consider the following differential equations:

A) Laplace equation $f_{xx} + f_{yy} = 0$
B) Wave equation $f_{xx} = f_{yy}$
C) Poisson equation $f_{xx} + f_{yy} = 4$
D) Heat equation $f_x = f_{yy}$
E) Transport equation $f_x = f_y$

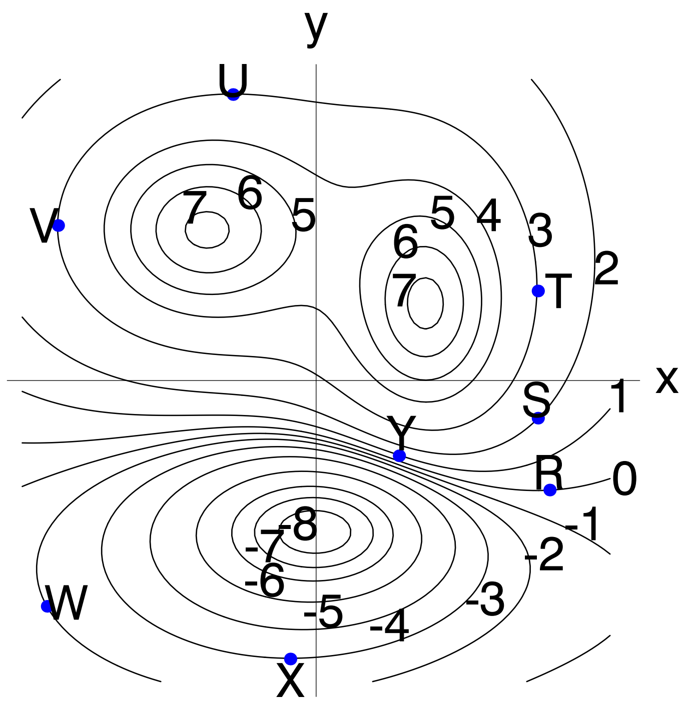
Given the functions $g(x,y) = \sin(x+y)$ and $h(x,y) = x^2 + y^2$. Which of the partial differential equations A,B,C,D,E do they satisfy?

Equation	g is a solution	g is not a solution	h is a solution	h is not a solution
A)				
B)				
C)				
D)				
E)				

Solution:

Equation	g is a solution	g is not a solution	h is a solution	h is not a solution
A)		x		x
B)	x		x	
C)		x	x	
D)		x		x
E)	x			x

Problem 4) (10 points)



a) (4 points) Circle the point at which the magnitude of the gradient vector ∇f is greatest. Mark exactly one point. Justify your answer.

R	S	T	U	V	W	X	Y

b) (3 points) Circle the points at which the partial derivative f_x is strictly positive. Mark any number of points on this question. Justify your answers.

R	S	T	U	V	W	X	Y

c) (3 points) We know that the directional derivative in the direction $(1,1)/\sqrt{2}$ is zero at one of the following points. Which one? Mark exactly one point on this question.

R	S	T	U	V	W	X	Y

Solution:

a) At the point Y the level curves are closest to each other indicating the steepest place and so the largest gradient.

b) At the points V and Y, the function increases, if we go into the x direction. In the other points, the function decreases, if we go into the x direction.

c) In order to have a zero directional derivative, we need the gradient to be zero or perpendicular into the direction $\vec{v}$. This is the case at the point S.

Problem 5) (10 points)

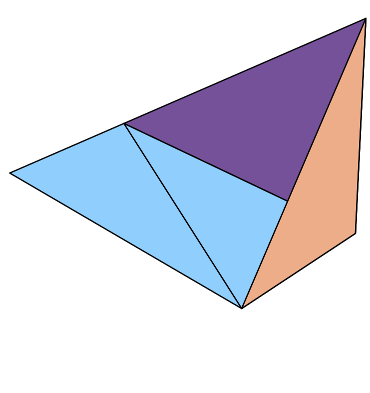
Find all the critical points of the function $f(x,y) = \frac{x^2}{2} + \frac{3y^2}{2} - xy^3$.
For each critical point, specify if it is a local maximum, a local minimum or a saddle point and show how you know.

Solution:
$\nabla f(x,y) = \langle x - y^3, 3y - 3xy^2 \rangle$. This is zero if $3y - 3y^5 = 0$ or $y(1-y^4) = 0$ which means $y = 0$ or $y = \pm 1$. In the case $y = 0$, we have $x = 0$. In the case $y = 1$, we have $x = 1$, in the case $y = -1$, we have $x = -1$. The critical points are $(0,0), (1,1), (-1,-1)$.
The discriminant is $f_{xx}f_{yy} - f_{xy}^2 = 3 - 6xy9y^4$. The entry f_{xx} is 1 everywhere.
Applying the second derivative test gives

Critical point	(0,0)	(1,1)	(-1,-1)
Discriminant	3	-12	-12
f_{xx}	1	1	1
Analysis	min	saddle	saddle

Problem 6) (10 points)

A beach wind protection is manufactured as follows. There is a rectangular floor $ACBD$ of length a and width b. A pole of height c is located at the corner C and perpendicular to the ground surface. The top point P of the pole forms with the corners A and C one triangle and with the corners B and C an other triangle. The total material has a fixed area of $g(a,b,c) = ab + ac/2 + bc/2 = 12$ square meters. For which dimensions a,b,c is the volume $f(a,b,c) = abc/6$ of the tetrahedral protected by this configuration maximal?



Solution:
The Lagrange equations are

$$\begin{aligned} bc &= \lambda(b+c/2) \\ ac &= \lambda(a+c/2) \\ ab &= \lambda(a+b)/2 \\ ab+bc/2+ac/2 &= 12 \ . \end{aligned}$$

Dividing the first to the second equation leads to $a=b$. Dividing the second to the third equation gives $c=2b$. Substituting a and c gives $b^2+b^2+b^2=12$ or $b=2$. Therefore $a=2, b=2, c=4$ is the optimal configuration. The maximal volume is $f(2,2,4)=8/3$.

Problem 7) (10 points)

A spaceship approaches its base $B=(0,0,-\pi/2)$ along the path

$$\mathbf{r}(t) = (\sin^2(t), 1-\cos(t), -\pi/2-t).$$

The base is protected by a force shield given by the equation $x^2+2y^2+z^2/\pi^2=3$. At time $t=-\pi/2$, the spaceship passes through the shield.

a) (5 points) At that time, does the ship pass through the shield at a right angle to the shield?

b) (5 points) The force shield is generated by a power station located at the point $(0,0,0)$. In the moment when the spaceship is passing through the shield, what is the rate of change of the distance from the spaceship to the power station?

Solution:

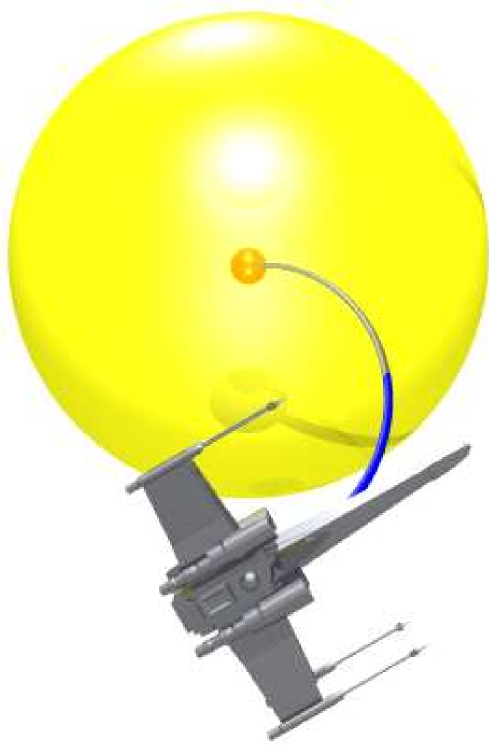
a) At the time $t = -\pi/2$, the spaceship is located at $(1, 1, 0)$. The velocity vector of the spaceship is $\langle 2\sin(t)\cos(t), \sin(t), -1\rangle$. At the time $t = -\pi/2$, the velocity vector is $\vec{v} = \langle 0, -1, -1\rangle$.

For $f(x, y, z) = x^2 + 2y^2 + z^2/\pi^2$, the gradient is $\nabla f = \langle 2x, 4y, 2z/\pi^2\rangle$. At the point $(1, 1, 0)$, the gradient vector is the vector $\vec{n} = \langle 2, 4, 0\rangle$, which is therefore perpendicular to the force shield at that point.

The spaceship passes through the shield at a right angle to the shield if and only if the velocity vector $\mathbf{v} = \langle 0, -1, -1\rangle$ is parallel to the normal vector $\mathbf{n} = \langle 2, 4, 0\rangle$. These vectors are clearly not parallel, so the spaceship does **not** pass through the shield at a right angle.

b) The distance from the origin is $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. The gradient of the distance function is $\nabla d = \langle x, y, z\rangle/\sqrt{x^2 + y^2 + z^2}$. At the point $(1, 1, 0)$, this vector is $\langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle$. The rate at which the distance is changing is the dot product of the gradient with the velocity vector of the spaceship by the chain rule. This is

$$\nabla d \cdot \vec{v} = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle \cdot \langle 0, -1, -1\rangle = -1/\sqrt{2}\ .$$



Problem 8) (10 points)

Given the function

$$f(x, y) = \sqrt{105 - 2x^2 - 3y^2}\ .$$

a) (4 points) Use the technique of linear approximation at the point $(1, 1)$ to estimate

$f(1.01, 0.9)$.

b) (3 points) Find a unit vector pointing in the direction at $(1,1)$ where the function decreases fastest.

c) (3 points) Find the tangent line to the curve $\sqrt{105 - 2x^2 - 3y^2} = 10$ at the point $(1,1)$.

Solution:
a) $L(x,y) = f(1,1) + f_x(1,1)(x - x_0) + f_y(1,1)(y - y_0)$.
$f(1,1) = 10$
$f_x(x,y) = -2x/f(x,y)$, $f_y(x,y) = -3y/f(x,y)$. $a = f_x(1,1) = -1/5$
$b = f_y(1,1) = -3/10$
$L(1.01, 0.9) = 10 - 1/5 \cdot 0.01 + 3/10 \cdot 0.1 = 10 - 0.002 + 0.03 = \boxed{10.028}$.
b) The direction in which the function decreases fastest is to $-\nabla f/||\nabla f||$ which is $\boxed{\langle 2,3\rangle/\sqrt{13}}$.
c) The tangent line has the form $2x + 3y = d$, where d can be obtained by plugging in the point $(1,1)$ which is $d = 5$. The equation of the tangent line is $\boxed{2x + 3y = 5}$.

Problem 9) (10 points)

Let S be the surface of revolution for which the distance r to the z-axis is $g(z) = e^z$.

a) (3 points) Find a parameterization of S.

b) (3 points) Find an implicit equation $f(x,y,z) = c$ which describes this surface.

c) (4 points) Find the tangent plane to S at the point $(-e, 0, 1)$.

Solution:
a) $\vec{r}(r,\theta) = \langle e^z \cos(\theta), e^z \sin(\theta), z\rangle$.
b) $r = e^z$ translates into $g(x,y,z) = x^2 + y^2 - e^{2z} = 0$. This is an implicit equation describing the surface.
c) The gradient of g is $\nabla g(x,y,z) = \langle 2x, 2y, -2e^{2z}\rangle$. The gradient of g at the point $(-e,0,1)$ is $\langle -2e, 0, -2e^2\rangle$. The tangent plane has the equation $-2ex - 2e^2y = d$. We find the value of d by plugging in $(x,y,z) = (-e,0,1)$. It is $d = 2e^2$. So, the equation is $-2ex - 2e^2z = 2e^2 - 2e^2 = 0$. The final result can be written as is $\boxed{x + ez = 0}$.