Math 192r, Problem Set #20
(due 12/11/01)

1. Let $f(\cdot)$ be some polynomial (in one variable), and define a sequence of rational functions $r_n = r_n(x, y)$ with the initial conditions $r_0 = x$, $r_1 = y$ and the recurrence $r_{n+2} = f(r_{n+1})/r_n$ ($n \geq 0$). Here x and y are formal indeterminates, so you don't need to worry about ill-definedness arising from a vanishing denominator.

   (a) Find an f such that the sequence of polynomials r_n is periodic with period 5.

   (b) Find an f (of degree at least 3, and with at least two terms) for which each of the rational functions $r_2, r_3, \ldots$ is a Laurent polynomial in x and y, such that the one-dimensional recurrence associated with f is a special case of a two-dimensional recurrence (analogous to frieze patterns or number walls) that also has the Laurentness property. (Proofs are not required for this part of the problem.)

   (c) Find a two-variable polynomial $f(\cdot, \cdot)$ that genuinely involves both of its variables such that the sequence of rational functions r_n with the initial conditions $r_0 = x$, $r_1 = y$, $r_2 = z$ and the recurrence $r_{n+3} = f(r_{n+1}, r_{n+2})/r_n$ is not periodic, and such that each r_n is a Laurent polynomial in x, y and z. (Proofs are not required for this problem.)

2. Given formal indeterminates $x_{i,j}$ and $y_{i,j}$, define $f(i, j, k)$ to be $x_{i,j}$ if $k = 0$ and $y_{i,j}$ if $k = 1$, and for $k > 1$ recursively define

$$f(i, j, k) = \frac{f(i-1, j-1, k-1)f(i+1, j+1, k-1) + f(i-1, j+1, k-1)f(i+1, j-1, k-1)}{f(i, j, k-2)}.$$

   (Note that this is Dodgson condensation with the minus-sign replaced by a plus-sign.)

(more)

(a) Submit code that demonstrates that $f(i,j,k)$ is a Laurent polynomial in the x- and y-variables for $k = 2, 3, 4$, and that all coefficients in this Laurent polynomial equal $+1$. (To say that code "demonstrates" the truth of a proposition, I don't mean that it generates output which a human could look over in order to convince herself/himself that the proposition is true. I mean that the code evaluates a boolean expression that encodes the proposition in question, and the proposition evaluates to `true`.)

(b) Give a conjectural pairing between the terms of the Laurent polynomial $f(i,j,k)$ and domino tilings of the Aztec diamond of order $k-1$, and verify it for $k \leq 3$.

(c) For $k \leq 6$, count how many terms there are in the Laurent polynomial obtained from $f(i,j,k)$ by replacing all the x-variables by 1. Repeat, this time instead replacing all the y-variables by 1.