Math 192r, Problem Set #17
(due 11/29/01)

1. Let $P(n)$ and $Q(n)$ denote the numerator and denominator obtained when the continued fraction

$$x_1 + (y_1/(x_2 + (y_2/(x_3 + (y_3/ \cdots + (y_{n-2}/(x_{n-1} + (y_{n-1}/x_n))) \cdots)))))$$

is expressed as an ordinary fraction. Thus $P(n)$ and $Q(n)$ are polynomials in the variables $x_1, ..., x_n$ and $y_1, ..., y_{n-1}$.

   (a) By examining small cases, give a conjectural bijection between the terms of the polynomial $P(n)$ and domino tilings of the 2-by-n rectangle, and a similar bijection between the terms of the polynomial $Q(n)$ and domino tilings of the 2-by-$(n-1)$ rectangle, as well as a conjecture that gives all the coefficients.

   (b) Prove your conjectures from part (a) by induction on n.

2. Let $R(n)$ denote the determinant of the n-by-n matrix M whose i, jth entry is equal to

$$\begin{cases} x_i & \text{if } j = i, \\ y_i & \text{if } j = i+1, \\ z_{i-1} & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) By examining small cases, give a conjectural bijection between the terms of the polynomial $R(n)$ and domino tilings of the 2-by-n rectangle, and a conjecture for the coefficients.

   (b) Prove your conjectures from part (a) by induction on n.

3. Consider a triangular array in which the top row is of length n, the next row is of length $n-1$, etc., with each row (other than the last) being centered above the row beneath. Whenever such an array contains four entries arranged like

(more)

$$\begin{array}{ccc} & w & \\ x & & y \\ & z & \end{array}$$

we'll say that these entries satisfy the diamond condition if $wz - xy = 1$. If the diamond condition is satisfied everywhere, we'll say that the array is a diamond pattern. Thus, for instance, the array

$$\begin{array}{ccccccc} a & & b & & c & & d \\ & e & & f & & g & \\ & & h & & i & & \\ & & & j & & & \end{array}$$

with a, b, c, d, e, f, g non-zero is a diamond pattern iff $h = (ef + 1)/b$, $i = (fg + 1)/c$, and $j = (hi + 1)/f$.

Note that if the top two rows of a diamond pattern contain no zeroes, there is a unique way to extend down. This is also true if the top two rows consist of distinct formal indeterminates. Let $D(x_1, x_3, \ldots, x_{2n+1}; y_2, y_4, \ldots, y_{2n})$ be the bottom entry of a diamond pattern whose first row is $x_1, x_3, \ldots, x_{2n+1}$ and whose second row is $y_2, y_4, \ldots, y_{2n}$. By examining small cases, you will find that $D(x_1, x_3, \ldots, x_{2n+1}; y_2, y_4, \ldots, y_{2n})$ can always be expressed as a multivariate Laurent polynomial. Give a conjectural bijection between the terms of this Laurent polynomial and domino tilings of the 2-by-$(2n-2)$ rectangle (for $n \geq 1$). Include also a conjecture governing the coefficients.

4. Repeat the problem, but with the diamond condition $ad - bc = 1$ replaced by the "frieze condition" $ad - bc = -1$. Let $F(x_1, x_3, \ldots, x_{2n+1}; y_2, y_4, \ldots, y_{2n})$ be the bottom entry of a frieze pattern whose first row is $x_1, x_3, \ldots, x_{2n+1}$ and whose second row is $y_2, y_4, \ldots, y_{2n}$. By examining small cases, you will find that $F(x_1, x_3, \ldots, x_{2n+1}; y_2, y_4, \ldots, y_{2n})$ can always be expressed as a multivariate Laurent polynomial. Give a conjectural bijection between the terms of this Laurent polynomial and domino tilings of the 2-by-$(2n-2)$ rectangle (for $n \geq 1$). Include also a conjecture governing the coefficients.